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The parametrix of the Laplace-Beltrami operator

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Abstract

We examine the general theory of pseudodifferential operators on flat Euclidean space and apply results to the Laplace-Beltrami operator Δ_g on a Riemannian manifold (M, g). In particular, we show that Δ_g on the manifold $(\mathbb{R}^n, g)$ is an elliptic pseudodifferential operator. Furthermore, we calculate the leading order terms of its parametrix. Using normal coordinates on Riemannian manifolds, we reduce the parametrix. Finally, in dimensions two and three we calculate the Schwartz kernels $K_{-2}(x, y)$ and $K_{-4}(x, y)$ of the leading order terms of this parametrix. Using this parametrix we show that if $\Delta_g u = f$ on $(\mathbb{R}^n, g)$, then $u = \int_{\mathbb{R}^n}(K_{-2}(x, x-y) + K_{-4}(x, x-y))f(y)\,\mathrm{d}y + h(x)$, where $h(x) \in C^1$ if the dimension $n = 3$ and $h(x) \in C^2$ if the dimension $n = 2$.

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Contents

Introduction

In an abstract sense, finding a solution to the Laplace (or Poisson) equation $\Delta u = f$ is easy. After we compute the Fourier transform, the equation turns into $-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$, and the solution to the original equation is found using the inverse Fourier transform, such that $u(x) = \mathcal{F}^{-1}[-\hat{f}/|\xi|^2]$. Similarly, for a linear partial differential operator $\mathbf{D}$ with constant coefficients, a solution to the partial differential equation $\mathbf{D}u = f$ can be found, in theory, with relative ease.

However once the coefficients of the linear partial differential operator $\mathbf{D}$ also become space dependent, this theoretical solution already breaks down. By defining the *symbol* of this operator $\mathbf{D}$, we may still use Fourier theory, however many "nice" properties are still lost. One such example is the composition of linear partial differential operators. The theory of *pseudodifferential operators* aims to solve these issues, by generalising the types of differential operators.

Moreover, the theory of pseudodifferential equations is necessary, once one looks at PDEs on smooth manifolds. Since for many manifolds we can only give local charts, we cannot even expect that the theory of constant coefficient partial differential equations suffices. This thesis gives an introduction to the theory of pseudodifferential equations, and gives some applications of it in the theory of partial differential equations on manifolds.

Chapters 1 and 2 give an introduction to the theory of pseudodifferential equations, and are based on books on pseudodifferential operators such as [Won14], and on distribution theory such as [FJ98] and use some functional analysis theory found in books such as [Con19].

In Chapter 1 we focus on distribution theory. We show that tempered distributions can be approximated by Schwartz functions. Furthermore, we define the Fourier transform, and use some elementary properties to compute the Fourier transform of some homogeneous distributions.

Chapter 2 gives the definition of a pseudodifferential operator, and discusses how one composes two pseudodifferential operators, and shows that finding an (approximate) inverse to a pseudodifferential operator is possible, if the operator is *elliptic*. Furthermore, we show that pseudodifferential operators are bounded linear operators between Sobolev spaces, and show that if a pseudodifferential operator T_a is elliptic, then the solution u to the pseudodifferential equation $T_a u = f$ is smooth whenever f is smooth.

Chapter 3 gives an introduction to the theory of pseudodifferential equations on smooth manifolds. It starts by defining an elliptic pseudodifferential operator on a Riemannian manifold, called the *Laplace-Beltrami operator* Δ_g. We explain the necessary material from Riemannian Geometry that can be found in more detail in [Lee18] and [Lee13], and using the machinery built up in the previous chapters, we then compute the approximate inverse of this Laplace-Beltrami operator, and give an integral operator representation. Using this integral representation, we finish by giving approximate solution to the pseudodifferential equation $\Delta_g u = f$.

Some final remarks before the mathematics starts. I have tried to prove every theorem, proposition, lemma and corollary in this thesis, for only very few do I refer the reader to a different source. I hope, therefore, that my fellow Master Mathematics students with some background in Fourier analysis can then follow the thesis from cover to cover. Finally, I have tried to keep any falsehoods and mistakes from entering this thesis, but I will never be completely certain that I have found and corrected all of those which did.

Notation convention

Notation will be explained where necessary, however for some quantities used throughout this thesis notation is given below

(a) A multi-index α is an element of $\mathbb{N}_0^n$, with length $|\alpha| = \sum_{j=1}^n \alpha_j$, order $m = \max_j \alpha_j$, factorial $\alpha! = \prod_{j=1}^n \alpha_j!$. The set of multi-indices has a natural partial ordering, such that $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \ldots, n$. If $\beta \leq \alpha$, then the binomial $\binom{\alpha}{\beta} = \prod_{j=1}^n \binom{\alpha_j}{\beta_j}$.

(b) The partial derivative $\partial/\partial x$ may be denoted by ∂_x, and the multidimensional partial derivative $\partial_x^\alpha = \prod_{j=1}^n \partial_{x_j}^{\alpha_j}$ in $\mathbb{R}^n$. Finally, $D_x^\alpha = (-i)^{|\alpha|}\partial_x^\alpha$.

(c) Δ will denote the Laplacian $\sum_{j=1}^n \partial_{x_j}^2$.

(d) If X is any topological vector space, then X' denotes its continuous dual space.

1 Preliminaries

In this chapter, some preliminaries are discussed such as distribution theory. Most information can be found in books such as [FJ98].

1.1 Linear partial differential equations

Pseudodifferential operators can be seen as a generalisation of constant coefficient linear partial differential operators. We give a definition of linear partial differential operators below.

Definition 1.1 (Linear partial differential operator)**.** On $\mathbb{R}^n$ a constant coefficient linear partial differential operator $\mathbf{D}$ of order m is a linear operator of the form

$$\mathbf{D} = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \ , \tag{1.1}$$

where α is a multi-index, and D^α is a derivative.

Definition 1.2 (Symbol)**.** For a linear partial differential operator $\mathbf{D}$ of order m, define its symbol $P(\mathbf{D})(\xi)$ by its Fourier transform. In formulaic terms

$$P(\mathbf{D})(\xi) = P_{\mathbf{D}}(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha \ . \tag{1.2}$$

The definitions above are used very widely, but can be extended to greater generality as done below.

Definition 1.3 (Linear partial differential operator)**.** On $\mathbb{R}^n$ a linear partial differential operator $\mathbf{D}$ of order m is a linear operator of the form

$$\mathbf{D}(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \ , \tag{1.3}$$

where $a_\alpha(x)$ may now depend on the variable x.

Definition 1.4 (Symbol)**.** For a linear partial differential operator $\mathbf{D}(x)$ of order m, define its symbol $P(\mathbf{D})(x, \xi)$ by its Fourier transform. In formulaic terms

$$P(\mathbf{D})(x, \xi) = P_{\mathbf{D}}(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \ . \tag{1.4}$$

We will see in the next sections what we mean by the Fourier transform of an operator. Even though it is not evident at this point, we will see that the definition is consistent.

1.2 Schwartz functions and Tempered distributions

In this section we look at the space of Schwartz functions, and its continuous dual space of tempered distributions. We show that the Schwartz functions are sequentially dense in the tempered distributions.

Definition 1.5 (Schwartz space)**.** On $\mathbb{R}^n$ there is a set of functions $\mathcal{S}(\mathbb{R}^n)$ given by all $f \in C^\infty(\mathbb{R}^n)$ such

$$\sup_{x\in\mathbb{R}^n} \left|x^\beta(D^\alpha f)\right| < \infty, \quad \text{for all multi-indices } \alpha, \beta. \tag{1.5}$$

Together with the semi-norms $p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\beta(D^\alpha f)|$ for all multi-indices α and β, the Schwartz space is a locally convex Fréchet space.

The first proposition shows that the Schwartz functions are also elements of the L^p-spaces. We will use this fact throughout this thesis.

Proposition 1.6 ($\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$)**.** For $p \in [1, \infty]$ there is a continuous inclusion of function spaces $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$.

Proof. Separate the cases $p = \infty$ and $p \neq \infty$. For $p = \infty$, the inclusion is obviously true, since $\sup_{x\in\mathbb{R}^n} |x^0(D^0 f)| = \|f\|_\infty < \infty$, whenever $f \in \mathcal{S}(\mathbb{R}^n)$.

For $p \in [1, \infty)$ notice that $g = f^p \in \mathcal{S}(\mathbb{R}^n)$ whenever $f \in \mathcal{S}(\mathbb{R}^n)$. Hence, it is sufficient to show that $f \in L^1(\mathbb{R}^n)$, whenever $f \in \mathcal{S}(\mathbb{R}^n)$.

Take $N > n$ an integer then $(1 + |x|^2)^{-N} \in L^1(\mathbb{R}^n)$. Indeed:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^N}\,\mathrm{d}x &= A_{n-1}\int_0^\infty \frac{r^{n-1}}{(1+r^2)^N}\,\mathrm{d}r \\ &\leq A_{n-1}\left(\int_0^1 1\,\mathrm{d}r + \int_1^\infty \frac{1}{(1+r^2)}\,\mathrm{d}r\right) < \infty\end{aligned} \tag{1.6}$$

where A_{n-1} is the (hyper-)surface of the $n-1$ Euclidean (hyper-)sphere S^{n-1} in $\mathbb{R}^n$. Thus using the fact that f is a Schwartz function there is some finite constant $C \geq 0$ such that

$$\sup_{x\in\mathbb{R}^n} |(1 + x^2)^N f(x)| = C\ . \tag{1.7}$$

Hence,

$$\int_{\mathbb{R}^n} |f(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}^n} \frac{C}{(1+|x|^2)^N}\,\mathrm{d}x < \infty \tag{1.8}$$

and $f \in L^1(\mathbb{R}^n)$. ■

In this thesis we will work mostly with tempered distributions.

Definition 1.7 (Tempered distributions)**.** Given the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$, define its continuous dual space $\mathcal{S}'(\mathbb{R}^n)$ with the weak-* topology. In other words a tempered distribution is a continuous linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$, and we say that a sequence of tempered distributions u_n converges to 0 if and only if for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ the sequence $\langle u_n, \phi \rangle$ converges to 0 in $\mathbb{R}$.

Sometimes tempered distributions are referred to as generalised functions. This is due to the fact that they may be approximated by a sequence of Schwartz functions.

Theorem 1.1 (Density of Schwartz functions in the tempered distributions)**.** *The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is sequentially dense in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ (with respect to the weak-* topology on $\mathcal{S}'(\mathbb{R}^n)$).*

We will prove this theorem in two steps: first we prove that $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ is dense for $p \in [1, \infty)$, and then, using the machinery built up, we will prove $\mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ is dense. For these theorema we need some preparations. In particular, we need the *convolution*.

Definition 1.8 (Convolution)**.** For functions $f, g \in L^1(\mathbb{R}^n)$ define the convolution operator by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y \ . \tag{1.9}$$

The following proposition shows some properties of the convolution operator.

Proposition 1.9. For $f, g, h \in L^1(\mathbb{R}^n)$ the following hold for the convolution.

(a) Associativity: $(f * g) * h = f * (g * h)$

(b) Distributivity: $f * (g + h) = f * g + f * h$

(c) Symmetry: $f * g = g * f$

Proof. (a) Writing out the definition we see

$$\begin{aligned}
((f * g) * h)(x) &= \int_{\mathbb{R}^n} (f * g)(x-y)h(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y-z)g(z)\,\mathrm{d}z\; h(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-w)g(w-y)\,\mathrm{d}w\; h(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} f(x-w)\int_{\mathbb{R}^n} g(w-y)h(y)\,\mathrm{d}y\;\; \mathrm{d}w \\
&= \int_{\mathbb{R}^n} f(x-w)(g * h)(w)\,\mathrm{d}w \\
&= (f * (g * h))(x) \ ,
\end{aligned} \tag{1.10}$$

by Fubini's theorem.

(b) Follows from the linearity of the integral.

(c) Follows by a change of variables. Indeed:

$$\begin{aligned}(f * g)(x) &= \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(w)g(x-w)\,\mathrm{d}w = (g * f)(x)\ .\end{aligned} \tag{1.11}$$

This completes the proof. ■

It is possible to define the convolution operator on a tempered distribution by the following prescription

$$\langle \psi * u, \phi \rangle = \left\langle u, \tilde{\psi} * \phi \right\rangle\ , \tag{1.12}$$

if we assume that $\psi \in C_0^\infty(\mathbb{R}^n)$ is a compactly supported smooth function, and $\tilde{\psi}(x) = \psi(-x)$. This prescription is also consistent, since if $u \in L^1(\mathbb{R}^n)$ then

$$\begin{aligned}\langle \psi * u, \phi \rangle &= \int_{\mathbb{R}^n} (\psi * u)(x)\phi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x-y)u(y)\,\mathrm{d}y\ \phi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x-y)\phi(x)\,\mathrm{d}x\ u(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} (\tilde{\psi} * \phi)(y)u(y)\,\mathrm{d}y = \left\langle u, \tilde{\psi} * \phi \right\rangle\ .\end{aligned} \tag{1.13}$$

Now take a compactly supported smooth function $\psi \in C_0^\infty(\mathbb{R}^n)$, with the properties that $0 \le \psi(x) \le 1$, and $\psi(-x) = \psi(x)$ for all $x \in \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} \psi(x)\,\mathrm{d}x = 1\ . \tag{1.14}$$

We will see in upcoming sections that such functions exists. For $\varepsilon > 0$ define ψ_ε by $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$, which satisfies

$$\int_{\mathbb{R}^n} \psi_\varepsilon(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \psi(x) = 1\ , \tag{1.15}$$

by simple change of variables. Furthermore, the following proposition holds, which is crucial for the proof of Theorem 1.1.

Proposition 1.10. Let $p \in [1, \infty)$ and take $f \in L^p(\mathbb{R}^n)$ a function. Then as ε converges to 0, the function $\psi_\varepsilon * f \in L^p$ converges to f.

Proof. We calculate in the p-norm the distance between $\psi_\varepsilon * f$ and f:

$$\begin{aligned}
\|\psi_\varepsilon * f - f\|_p &= \left[\int_{\mathbb{R}^n} |\psi_\varepsilon * f - f|^p\right]^{1/p} \\
&= \left[\int_{\mathbb{R}^n} |f * \psi_\varepsilon - f|^p\right]^{1/p} \\
&= \left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} f(x-y)\psi_\varepsilon(y)\,\mathrm{d}y - f(x)\right|^p \mathrm{d}x\right]^{1/p} \\
&= \left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} (f(x-y) - f(x))\psi_\varepsilon(y)\,\mathrm{d}y\right|^p \mathrm{d}x\right]^{1/p} \\
&= \left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} (f(x-\varepsilon\hat{y}) - f(x))\psi(\hat{y})\,\mathrm{d}\hat{y}\right|^p \mathrm{d}x\right]^{1/p} .
\end{aligned} \tag{1.16}$$

Therefore, by Minkowski's inequality it follows that

$$\begin{aligned}
\|\psi_\varepsilon * f - f\|_p &\leq \int_{\mathbb{R}^n} |\psi(\hat{y})| \left(\int_{\mathbb{R}^n} |f(x-\varepsilon\hat{y}) - f(x)|^p\right)^{1/p} \mathrm{d}\hat{y} \\
&= \int_{\mathbb{R}^n} |\psi(\hat{y})| \|f_{-\varepsilon\hat{y}} - f\|_p \,\mathrm{d}\hat{y} ,
\end{aligned} \tag{1.17}$$

where f_y denotes the translated function $f(x+y)$. Now using the fact that the p-norm is continuous under translations, it follows that for all $\hat{y}$ the distance between f and $f_{-\varepsilon\hat{y}}$ converges to 0 as ε converges to 0. By the dominated convergence theorem, it follows that

$$\lim_{\varepsilon\to 0} \|\psi_\varepsilon * f - f\|_p \leq \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} |\psi(\hat{y})| \|f_{-\varepsilon\hat{y}} - f\|_p \,\mathrm{d}\hat{y} = 0 , \tag{1.18}$$

which proves the proposition. ∎

Lemma 1.11. If $f \in C_0(\mathbb{R}^n)$ is a compactly supported continuous function, and $g \in C_0^\infty(\mathbb{R}^n)$ is a compactly supported smooth function, then

$$\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g) \text{ and } \partial_x^\alpha (f * g)(x) = (f * (\partial_x^\alpha g))(x) . \tag{1.19}$$

Hence, $f * g$ is a compactly supported smooth function.

Proof. Let $x \in \mathbb{R}^n$ be such that $(f * g)(x) \neq 0$, then

$$0 \neq \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y \tag{1.20}$$

hence there must be a point $y \in \mathbb{R}^n$ such that $f(x-y)g(y) \neq 0$. Hence, we have $x = (x-y) + y \in \operatorname{supp}(f) + \operatorname{supp}(g)$.

Suppose now that $x \in \operatorname{supp}(f * g)$ such that $(f * g)(x) = 0$, then for any $\varepsilon > 0$, the ball $B_{\varepsilon,x}$ of radius ε around x contains a point x' such that $(f * g)(x') \neq 0$. Then there is a y such that $f(x'-y)g(y) \neq 0$, such that $x = x' + tv = (x'-y) + y + tv$ for some vector v and some scalar $t \in (0, \varepsilon)$. Since for any $\varepsilon > 0$ there is such an x' it follows that $x \in \operatorname{supp}(f) + \operatorname{supp}(g)$.

Now to prove that $f*g$ is smooth, notice that it is sufficient to prove that, for any i, the partial derivative $\partial_{x_i}(f*g)(x)$ exists, and that $\partial_{x_i}(f*g)(x) = (f*(\partial_{x_i}g))(x)$, since then $h = \partial_{x_i}g$ is also smooth. This is indeed the case by the dominated convergence theorem, because both f, g are compactly supported, hence

$$\begin{aligned}\partial_{x_i}(f*g)(x) &= \partial_{x_i}\int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y \\ &= \partial_{x_i}\int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \partial_{x_i}g(x-y)f(y)\,\mathrm{d}y \\ &= ((\partial_{x_i}g)*f)(x) = (f*(\partial_{x_i}g))(x)\ .\end{aligned} \tag{1.21}$$

This proves that $(f*g)$ is a compactly supported smooth function. ■

We can now prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Proposition 1.12. For $p \in [1, \infty)$, the space $C_0^\infty(\mathbb{R}^n)$ of compactly supported smooth functions is dense in $L^p(\mathbb{R}^n)$.

Proof. The proof of this proposition will make use of the transitivity of dense subspaces. In particular, we use that the simple functions are dense in $L^p(\mathbb{R}^n)$ to show that the compactly supported continuous functions are dense in $L^p(\mathbb{R}^n)$. Then we show that any compactly supported continuous function can be approximated by a compactly supported smooth function.

Take a measurable set A, with finite measure. We will show that there is a compactly supported continuous function f which approximates the indicator function on A. Take $\varepsilon > 0$,then because the Lebesgue measure is both inner and outer regular, there is a measurable open set U and a compact set K such that $K \subseteq A \subseteq U$ and $\mu(U \setminus K) = \mu(U) - \mu(K) < \varepsilon$. Now by Uhrysohn's lemma there is a continuous function $0 \leq f \leq 1$ with $\operatorname{supp} f \subseteq U$ such that $f|_K = 1$. Then

$$\int_{\mathbb{R}^n} |\chi_A - f|^p\,\mathrm{d}\mu = \int_{U\setminus K} |\chi_A - f|^p\,\mathrm{d}\mu < \varepsilon. \tag{1.22}$$

Now by taking finite linear combinations of indicator functions we obtain the simple functions, hence the space of compactly supported continuous functions $C_0(\mathbb{R}^n)$ is dense in the space of simple functions. Because the space of simple functions is dense in $L^p(\mathbb{R}^n)$ it follows that $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. For a more detailed proof see [Rud87] page 69.

Next notice that if $f \in C_0(\mathbb{R}^n)$ and $g \in C_0^\infty(\mathbb{R}^n)$, then it follows for the convolution $f*g$ that $\operatorname{supp} f*g \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$, and that $f*g$ is smooth, hence

$f * g \in C_0^\infty(\mathbb{R}^n)$. Thus, by using ψ_ε from before, it follows that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Hence, the proposition is proved using the inclusion of function spaces given by $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$. ■

Remark 1.13. The proof for $p = \infty$ falls apart because the compactly supported continuous functions $C_0(\mathbb{R}^n)$ is not dense in $L^\infty(\mathbb{R}^n)$. A counterexample is the constant 1 function, for which the approximation given in the proof does not converge in the ∞-norm.

For the full proof of Theorem 1.1 we will do the same steps for a tempered distribution. First we prove that compactly supported tempered distributions approximate general tempered distributions.

Lemma 1.14. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, and if $0 \le \chi \le 1$ is a compactly supported smooth function which is one on a neighbourhood of 0, and if we set $\chi_m(x) = \chi(x/m)$, then $u_m := \chi_m u$ converges to u in the weak topology on $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Take any $\phi \in \mathcal{S}(\mathbb{R}^n)$, then for the distribution $(1 - \chi_m)u$ it holds that

$$\langle u - u_m, \phi\rangle = \langle (1-\chi_m)u, \phi\rangle = \langle u, (1-\chi_m)\phi\rangle \xrightarrow{m\to\infty} 0 \ , \tag{1.23}$$

if $(1 - \chi_m)\phi$ converges to 0 in all semi-norms on $\mathcal{S}(\mathbb{R}^n)$. We calculate for all α, β

$$\begin{aligned} &|x^\alpha D_x^\beta (1-\chi_m(x))\phi(x)| \\ &\quad = \left| x^\alpha \sum_{\beta' \le \beta} \binom{\beta}{\beta'} D_x^{\beta-\beta'}(1-\chi_m(x))(D_x^{\beta'}\phi)(x) \right| \\ &\quad = \Bigg| x^\alpha (1-\chi_m(x))(D_x^\beta \phi)(x) \\ &\qquad\quad + x^\alpha \sum_{\beta'<\beta} \binom{\beta}{\beta'} m^{-|\beta-\beta'|} (D_x^{\beta-\beta'}\chi)(x/m)(D_x^{\beta'}\phi)(x) \Bigg| \\ &\quad \le \left| x^\alpha (1-\chi_m(x))(D_x^\beta\phi)(x)\right| \\ &\qquad\quad + \sum_{\beta'<\beta} \binom{\beta}{\beta'} m^{-|\beta-\beta'|} \left| x^\alpha (D_x^{\beta-\beta'}\chi)(x/m)(D_x^{\beta'}\phi)(x)\right| \ . \end{aligned} \tag{1.24}$$

For each $\varepsilon > 0$ and each $\beta' < \beta$ we can choose an $m_{\beta'}$ such that for all $m > m_{\beta'}$ we have

$$\binom{\beta}{\beta'} m^{-|\beta-\beta'|} \sup_{x\in\mathbb{R}^n} \left| x^\alpha (D_x^{\beta-\beta'}\chi)(x/m)(D_x^{\beta'}\phi)(x)\right| < \varepsilon/|\beta|^n \ . \tag{1.25}$$

Similarly, we can choose an m_β such that for all $m > m_\beta$ we have

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha (1-\chi_m(x))(D_x^\beta\phi)(x)\right| < \varepsilon/|\beta|^n \ . \tag{1.26}$$

Now by setting $M = \max_{\beta' \leq \beta}\{m_{\beta'}\}$, we have that for all $m > M$ that

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha D_x^\beta (1 - \chi_m)\phi \right| < \varepsilon \ , \tag{1.27}$$

which proves the lemma. ■

Corollary 1.15. Compactly supported tempered distributions are dense in the space of all tempered distributions.

Lemma 1.16. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a compactly supported tempered distribution and $\psi \in C_0^\infty(\mathbb{R}^n)$ is a compactly supported smooth function, then

$$(u * \psi)(x) \tag{1.28}$$

exists in the classical sense, is compactly supported and is smooth in the classical sense.

Proof. Because u and ψ are compactly supported, the prescription from Equation (1.12) applied to the compactly supported smooth function ϕ, which is 1 on some compact set $K \supseteq \mathrm{supp}(f) + \mathrm{supp}(g)$, results in the fact that

$$(u * \psi)(x) \tag{1.29}$$

exists in a classical sense. Furthermore, the calculations from Lemma 1.11 still hold, and yield that $(u * \psi)(x)$ is a compactly supported smooth function. ■

Using the previous lemmata we are now able to prove Theorem 1.1.

Proof. We will approximate a compactly supported tempered distribution by a compactly supported smooth function. Then using the fact that compactly supported tempered distributions are dense in the space of all tempered distributions, the theorem follows.

Take $u \in \mathcal{S}'(\mathbb{R}^n)$ a compactly supported tempered distribution, and ψ_ε as in Proposition 1.10, it follows that $(u * \psi_\varepsilon)$ is compactly supported and smooth by Lemma 1.16. Furthermore, it follows that $(\psi_\varepsilon * u)$ converges to u in the weak topology.

Hence, $C_0^\infty(\mathbb{R}^n)$ is dense in the compactly supported distributions, which in turn is dense in the entire space of tempered distributions. This shows that the inclusion of function spaces $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ are sequentially dense. ■

1.3 Fourier transform

This section is dedicated to the Fourier transform. We give a definition of the Fourier transform, give elementary properties, and define the inverse Fourier transform. Whole books can be written about the Fourier transform, so only the necessary properties and facts are given.

Definition 1.17 (Fourier transform on $\mathcal{S}(\mathbb{R}^n)$)**.** For a function $f \in \mathcal{S}(\mathbb{R}^n)$, define its Fourier transform by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x \ . \tag{1.30}$$

Notice that this integral is absolutely converging, since $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$.

Proposition 1.18 (Properties of the Fourier transform)**.** The Fourier transform is an operator with the following properties:

(a) Linearity: $\mathcal{F}[\lambda f + \mu g] = \lambda\mathcal{F}[f] + \mu\mathcal{F}[g]$,

(b) Products and derivatives: $\mathcal{F}[x^\beta D_x^\alpha f] = (-1)^{|\beta|}\xi^\alpha D_\xi^\beta \mathcal{F}[f](\xi)$,

(c) Convolution: $\mathcal{F}[f * g] = (2\pi)^{n/2}\hat{f}\cdot\hat{g}$,

(d) Translation and unitary multiplication: $\mathcal{F}[f(x+y)](\xi) = e^{iy\cdot\xi}\mathcal{F}[f(x)](\xi)$ and $\mathcal{F}[e^{ix\cdot y}f(x)](\xi) = \mathcal{F}[f(x)](\xi - y)$,

(e) Dilation: $\mathcal{F}[f(tx)] = t^{-n}\mathcal{F}[f](t^{-1}\xi)$ for $t > 0$,

(f) Orthogonal invariance: Assume that $f(Ax) = f(x)$ for all $A \in \mathcal{O}(n)$, then $\hat{f}(A\xi) = \hat{f}(\xi)$,

(g) The Fourier transform is formally self-adjoint: $\int_{\mathbb{R}^n} \hat{f}g = \int_{\mathbb{R}^n} f\hat{g}$,

(h) The linear operator $\mathcal{F}^{-1}[f](\xi) = \check{f} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{+ix\cdot\xi} f(x)\,\mathrm{d}x$ provides an inverse to the Fourier transform,

(i) Plancherel-Parseval identity: $\|f\|_2 = \|\hat{f}\|_2$,

for all functions $f, g \in \mathcal{S}(\mathbb{R}^n)$, all scalars $\lambda, \mu \in \mathbb{C}$ and all multi-indices α, β.

Proof. (a) Linearity follows by the linearity of the integral

(b) By partial integration it follows that

$$\begin{aligned}
\mathcal{F}[D_{x_j} f](\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} D_{x_j} f(x)\,\mathrm{d}x \\
&= -\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(-i\partial_{x_j} e^{-ix\cdot\xi}\right) f(x)\,\mathrm{d}x \\
&= \frac{\xi_j}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x \\
&= \xi_j \cdot \hat{f}(\xi)
\end{aligned} \tag{1.31}$$

and by the dominated Convergence Theorem it follows that

$$\begin{aligned}\mathcal{F}[x_j \cdot f](\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} x_j f(x)\,\mathrm{d}x \\ &= -\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(i\partial_{\xi_j} e^{-ix\cdot\xi}\right) f(x)\,\mathrm{d}x \\ &= -D_{\xi_j}\left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x\right) \\ &= -D_{\xi_j}\hat{f}(\xi)\ .\end{aligned} \tag{1.32}$$

From (1.31) and (1.32) the formula $\mathcal{F}[x^\beta D_x^\alpha f] = (-1)^{|\beta|}\xi^\alpha D_\xi^\beta \mathcal{F}[f](\xi)$ follows.

(c) By Fubini it follows that

$$\begin{aligned}\mathcal{F}[f * g](\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x-y)\cdot\xi} e^{-iy\cdot\xi} \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} g(y) \int_{\mathbb{R}^n} e^{-i(x-y)\cdot\xi} f(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{(2\pi)^{n/2}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} g(y)\,\mathrm{d}y \int_{\mathbb{R}^n} e^{-i(z)\cdot\xi} f(z)\,\mathrm{d}z \\ &= (2\pi)^{n/2}\hat{f}(\xi)\cdot\hat{g}(\xi)\ .\end{aligned} \tag{1.33}$$

(d) Simple calculations show

$$\begin{aligned}\mathcal{F}[f(x+y)](\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x+y)\,\mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iy\cdot\xi} e^{-i(x+y)\cdot\xi} f(x+y)\,\mathrm{d}x \\ &= e^{iy\cdot\xi}\mathcal{F}[f(x)](\xi)\end{aligned} \tag{1.34}$$

and

$$\begin{aligned}\mathcal{F}[e^{ix\cdot y} f(x)](\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot(\xi-y)} f(x)\,\mathrm{d}x \\ &= \mathcal{F}[f(x)](\xi-y)\ .\end{aligned} \tag{1.35}$$

(e) By change of variables

$$\begin{aligned}\mathcal{F}[f(tx)] &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(tx)\,\mathrm{d}x \\ &= \frac{t^{-n}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iy\cdot(t^{-1}\xi)} f(y)\,\mathrm{d}y \\ &= t^{-n}\mathcal{F}[f](t^{-1}\xi)\ .\end{aligned} \tag{1.36}$$

(f) By change of variables

$$\begin{aligned}\hat{f}(A\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot A\xi} f(x)\,\mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iA^T x\cdot\xi} f(x)\,\mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(Ay)|\det A|\,\mathrm{d}y \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\,\mathrm{d}y = \hat{f}(\xi)\ .\end{aligned} \tag{1.37}$$

(g) By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f}(x)g(x)\,\mathrm{d}x &= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot x} f(y)\,\mathrm{d}y g(x)\,\mathrm{d}x \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot x} g(x)\,\mathrm{d}x f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(y)\hat{g}(y)\,\mathrm{d}y\ .\end{aligned} \tag{1.38}$$

(h) For $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(\hat{f})^\vee(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\,\mathrm{d}\xi\ . \tag{1.39}$$

Define I_ε by

$$I_\varepsilon = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi - \varepsilon^2|\xi|^2/2} \hat{f}(\xi)\,\mathrm{d}\xi\ . \tag{1.40}$$

By the dominated convergence theorem I_ε converges to $(\hat{f})^\vee(x)$ pointwise as $\varepsilon \to 0$. So it satisfies to show that I_ε converges to f pointwise.

Simple calculations show that the Fourier transform of $\phi(x) = e^{-|x|^2/2}$ is $\phi(\xi)$. Hence, if we set

$$g(\xi) = e^{ix\cdot\xi - \varepsilon^2|\xi|^2/2} \tag{1.41}$$

part (d) implies that

$$\hat{g}(\eta) = \varepsilon^{-n} e^{-|x-\eta|^2/(2\varepsilon^2)}\ . \tag{1.42}$$

Thus, by formal self-adjointness of the Fourier transform, it follows that

$$\begin{aligned}I_\varepsilon(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(\xi)\hat{f}(\xi)\,\mathrm{d}\xi = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{g}(\xi) f(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varepsilon^{-n} e^{-|x-\eta|^2/(2\varepsilon^2)} f(\xi)\,\mathrm{d}\xi \\ &= f * ((2\pi)^{-n/2}\phi_\varepsilon) \to f\end{aligned} \tag{1.43}$$

as $\varepsilon \to 0$ by Proposition 1.10. Hence, I_ε converges to f pointwise, and hence it follows that $\mathcal{F}^{-1}$ is an inverse to the Fourier transform.

(i) Using the density of the Schwartz-functions we can define the Fourier transform on L^p by taking an approximating sequence of functions if the Fourier transform is a bounded linear transform in the p-norm for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. This is indeed the case for $p = 2$. First define $\psi(x) = \overline{\varphi(-x)}$, then by a change of variables

$$\begin{aligned}\hat{\psi}(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\psi(x)\,\mathrm{d}x = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\overline{\varphi(-x)}\,\mathrm{d}x \\ &= \overline{(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,\mathrm{d}x} = \overline{\hat{\varphi}}\ .\end{aligned} \tag{1.44}$$

Thus,

$$\begin{aligned}\|\hat{\varphi}\|_2^2 &= \int_{\mathbb{R}^n} \varphi\overline{\varphi} = \int_{\mathbb{R}^n} \varphi(x)\psi(-x)\,\mathrm{d}x \\ &= (\varphi * \psi)(0) = ((\varphi * \psi)^\wedge)^\vee(0) \\ &= \int_{\mathbb{R}^n} (\varphi * \psi)^\wedge(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\hat{\psi}(\xi)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\overline{\hat{\varphi}(\xi)}\,\mathrm{d}\xi = \|\hat{\varphi}\|_2^2\ ,\end{aligned} \tag{1.45}$$

which proves that the Fourier transform extends to a bounded linear operator on L^2, and is isometric on it.

This finishes the proofs to this proposition. ∎

Definition 1.19. For a general tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ define its Fourier transform by the following prescription

$$\hat{u}(\varphi) = u(\hat{\varphi}) \qquad \varphi \in \mathcal{S}(\mathbb{R}^n)\ . \tag{1.46}$$

The following theorem gives an estimate for the Fourier transform of a compactly supported smooth tempered distribution.

Theorem 1.2 (Paley-Wiener)**.** *Suppose that $u \in \mathcal{S}'(\mathbb{R}^n)$, and suppose furthermore that there exists some constant $a \geq 0$ such that* $\operatorname{supp} u \subseteq \{|x| \leq a\}$. *Then the Fourier transform $\hat{u}$ of u is smooth and analytic. Furthermore, if u is smooth then for any $m = 0, 1, \ldots$ there is a constant $C_m \geq 0$ such that*

$$|\hat{u}(\xi)| \leq C_m(1 + |\xi|)^{-m}\ . \tag{1.47}$$

Proof. Because u is compactly supported we may see the Fourier transform of u as

$$\hat{u} = (2\pi)^{-n/2}u(e^{-ix\cdot\xi}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,\mathrm{d}x\ . \tag{1.48}$$

To see this apply u to the Fourier transform of a compactly supported smooth function φ for which $\varphi \equiv 1$ on the support of u, and apply Fubini's theorem. Hence, the Fourier-Laplace transform defined by

$$\hat{U}(\xi + i\eta) = \frac{1}{(2\pi)^{n/2}} \int_{|x|\leq a} e^{-ix\cdot(\xi+i\eta)}u(x)\,\mathrm{d}x \tag{1.49}$$

exists and satisfies $\hat{u}(\xi) = \hat{U}(\xi + i0)$. Then by the dominated convergence theorem, for any $j = 1, 2 \ldots, n$,

$$\partial_{\xi_j} - i\partial_{\eta_j}\hat{U} = \frac{1}{(2\pi)^{n/2}} \int_{|x|\leq a} (\partial_{\xi_j} - i\partial_{\eta_j}) e^{-ix\cdot(\xi+i\eta)} u(x)\,\mathrm{d}x = 0 \ . \tag{1.50}$$

Hence, $\hat{U}(\xi + i\eta)$ is holomorphic in all its variables, and thus analytic. It follows then that $\hat{u}(\xi)$ is smooth and analytic.

Now assume that u is also smooth, then u is in fact also a Schwartz function and the previous proposition shows that

$$|\xi^\alpha \hat{u}(\xi)| \leq \frac{1}{(2\pi)^{n/2}} \int_{|x|\leq a} |D_x^\alpha u(x)|\,\mathrm{d}x \leq V_a \sup_{|x|\leq a} |D_x^\alpha u(x)| < \infty, \tag{1.51}$$

where V_a is the volume of the ball of radius a. Now Equation (1.51) implies Equation (1.47). ■

1.4 Homogeneous distributions

In this section we define homogeneous distributions as in [Tay11]. The aim of this section is to compute the Fourier transform of the distributions $1/|x|^m$ for different integer values of m and in different dimensions n. But first we define the singular support of a distribution.

Definition 1.20 (Singular support)**.** Let $u \in \mathcal{S}'(\mathbb{R}^n)$, we say that u is smooth on an open set $\Omega \subseteq \mathbb{R}^n$, if there is a function $v \in C^\infty(\Omega)$ such that $u = v$ on Ω. Finally, the singular support of u is the smallest closed set K such that u is smooth on $\Omega = \mathbb{R}^n \setminus K$. We denote this singular support K by $\operatorname{sing\,supp} u$.

The following shows a use of the singular support of a distribution.

Definition 1.21 (Homogeneous distributions)**.** Let $f \in \mathcal{S}(\mathbb{R}^n)$. We say that f is homogeneous of degree $m \in \mathbb{C}$ if

$$D(t)f(x) = f(tx) = t^m f(x) \quad \text{for all } t > 0 \ . \tag{1.52}$$

By a change of variables it is clear that

$$\int_{\mathbb{R}^n} (D(t)f(x))g(x)\,\mathrm{d}x = t^{-n} \int_{\mathbb{R}^n} f(x)(D(t^{-1})g(x))\,\mathrm{d}x \ . \tag{1.53}$$

Hence, we say that $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree m if for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$\langle f, D(t)u\rangle = t^{-n} \left\langle D(t^{-1})f, u\right\rangle = t^m \langle f, u\rangle \quad \text{for all } t > 0 \ . \tag{1.54}$$

We say that any such distribution u which satisfies this property is an element of $\mathcal{H}_m(\mathbb{R}^n)$.

Example 1.22. The delta-distribution $\delta \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree $-n$. This is clear, since

$$\langle f, D(t)\delta \rangle = t^{-n} \left\langle D(t^{-1})f, \delta \right\rangle = t^{-n} f(0) = t^{-n} \langle f, \delta \rangle \ . \tag{1.55}$$

By Proposition 1.18 part (e), it follows that

$$\mathcal{F}[D(t)f] = t^{-n}(D(t^{-1})\mathcal{F}(f)) \ . \tag{1.56}$$

Hence, the Fourier transform is a linear map

$$\mathcal{F} : \mathcal{H}_m(\mathbb{R}^n) \to \mathcal{H}_{-m-n}(\mathbb{R}^n) \ . \tag{1.57}$$

Define the set $\mathcal{H}_m^{\#}(\mathbb{R}^n)$ as the intersection of distributions which are smooth outside the origin, i.e. have singular support given by the set $K = \{0\}$ and distributions which are homogeneous of degree m. Then

Lemma 1.23. The Fourier transform $\mathcal{F}$ is a linear map

$$\mathcal{F} : \mathcal{H}_m^{\#} \to \mathcal{H}_{-m-n}^{\#} \tag{1.58}$$

Proof. Clearly from the previous discussion it follows that homogeneity is preserved. It is left to prove that $\mathcal{F}$ preserves smoothness outside the origin.

Take some $u \in \mathcal{H}_m^{\#}$ and take some $0 \le \phi \le 1 \in C_0^\infty(\mathbb{R}^n)$, which is 1 on the set $\{|x| < R\}$. Then $u = \phi u + (1-\phi)u = u_1 + u_2$. By the Paley-Wiener Theorem 1.2, it follows that $\mathcal{F}[u_1]$ is smooth. Thus, the lemma is proven, once $\mathcal{F}[u_2]$ is smooth outside the origin.

The following lemma will imply this fact.

Lemma 1.24. Let $\mathcal{S}_1^m \subseteq \mathcal{S}'(\mathbb{R}^n)$ be the class of smooth functions, such that $|D_x^\alpha u(x)| \le C_\alpha (1+|x|^2)^{(m-|\alpha|)/2} = \langle x \rangle^{m-|\alpha|}$. For any $u \in \mathcal{S}_1^m(\mathbb{R}^n)$ it holds that $\hat{u} \in C^\infty(\mathbb{R}^n \setminus 0)$.

Proof. We will show that for any $u \in \mathcal{S}_1^m$ the function $x^\beta \hat{u}$ is bounded and continuous, and that all of its derivatives are too. For $m < -n$, it holds that any function $u \in \mathcal{S}_1^m$ is absolutely integrable, hence it follows that the Fourier transform is a continuous map

$$\mathcal{F} : \mathcal{S}_1^m \to L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \ . \tag{1.59}$$

Now take m to be arbitrary, then for $u \in \mathcal{S}_1^m$ it holds that $x^\alpha D^\beta u \in \mathcal{S}_1^{m+|\alpha|-|\beta|}$ hence

$$D_\alpha(x^\beta \hat{u}) = \mathcal{F}[D^\beta u] \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \tag{1.60}$$

if $|\beta| > m + n + |\alpha|$. This shows that $\hat{u}$ is smooth outside the origin. ■

We finish the proof of Lemma 1.23 by the observation that $u_2 \in \mathcal{S}_1^{\mathrm{Re}(m)}$, since on the set $\{|x| < R\}$ we have $u_2(x) = 0$ and outside that compact set we have that $|u_2| \le |u| \le (1+|x|^2)^{m/2}$ because u is homogeneous of degree m. ■

Theorem 1.3. *The Fourier transform of the distribution* $u(x) = 1/|x|^2$ *is given by*

$$\hat{u}(\xi) = \begin{cases} \log|\xi| & \text{if } n = 2, \\ 2^{(n-4)/2}\Gamma\left(\frac{n}{2} - 1\right)|\xi|^{2-n} & \text{if } n \geq 3 \end{cases} . \tag{1.61}$$

Proof. Unfortunately, as our distribution $u = 1/|x|^2$ is not locally integrable on $\mathbb{R}^2$, computing its Fourier transform is not immediately possible. The idea of the following paragraphs will be to create an extension E_φ of our homogeneous tempered distribution $u \in \mathcal{H}^{\#}_{-2}(\mathbb{R}^2) \subseteq \mathcal{S}'(\mathbb{R}^2)$ such that $\mathcal{F}[u] = \mathcal{F}[E_\varphi]$.

We define this $E_\varphi \in \mathcal{S}'(\mathbb{R}^n)$ by the following prescription

$$\langle v, E_\varphi u\rangle = \int_{\mathbb{R}^2} u(x)[v(x) - v(0)\varphi(x)]\,\mathrm{d}x \quad v \in \mathcal{S}(\mathbb{R}^n) , \tag{1.62}$$

where $\varphi(x) \in \mathcal{S}(\mathbb{R}^2)$ is any radial function, which satisfies $\varphi(0) = 1$. Now E_φ is no longer a homogeneous distribution, indeed:

$$\langle v, D(t)E_\varphi u\rangle = t^{-2}\int_{\mathbb{R}^2} u(x)[v(x) - v(0)\varphi(tx)]\,\mathrm{d}x = t^{-2}\left\langle v, E_{D(t)\varphi}u\right\rangle , \tag{1.63}$$

so that $D(t)E_\varphi u = t^{-2}E_{D(t)\varphi}u$. Notice furthermore, that $E_\varphi u$ is identical to u on $\mathbb{R}^2 \setminus 0$. This is due to the fact that the prescription of $E_\varphi u$ can be represented informally as

$$E_\varphi u = u(x)[1 - \varphi(x)\delta(x)] . \tag{1.64}$$

Furthermore, the dependence on the radial function can be seen from the following

$$\begin{aligned} \langle v, E_\varphi u - E_\psi u\rangle &= \int_{\mathbb{R}^2} u(x)[\psi(x) - \varphi(x)]v(0)\,\mathrm{d}x \\ &= \left(\int_{\mathbb{R}^2} u(x)[\psi(x) - \varphi(x)]\,\mathrm{d}x\right)\langle v, \delta\rangle . \end{aligned} \tag{1.65}$$

Hence by Equations (1.63) and (1.65), it follows that

$$\begin{aligned} D(t)E_\varphi u &= t^{-2}[E_{D(t)\varphi}u - E_\varphi u + E_\varphi u] \\ &= t^{-2}\left(\int_{\mathbb{R}^2} u(x)[\varphi(x) - \varphi(tx)]\,\mathrm{d}x\right)\delta + t^{-2}E_\varphi u \\ &= 2\pi t^{-2}\left(\int_0^\infty \frac{1}{r}[\varphi(r) - \varphi(tr)]\,\mathrm{d}r\right)\delta + t^{-2}E_\varphi u \\ &= 2\pi t^{-2}\left(\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty \frac{1}{r}\varphi(r)\,\mathrm{d}r - \int_\varepsilon^\infty \frac{1}{r}\varphi(tr)\,\mathrm{d}r\right)\delta + t^{-2}E_\varphi u \\ &= 2\pi t^{-2}\left(\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty \frac{1}{r}\varphi(r)\,\mathrm{d}r - \int_{t\varepsilon}^\infty \frac{1}{r}\varphi(r)\,\mathrm{d}r\right)\delta + t^{-2}E_\varphi u \\ &= 2\pi t^{-2}\left(\lim_{\varepsilon\downarrow 0}\int_\varepsilon^{t\varepsilon} \frac{1}{r}\varphi(r)\,\mathrm{d}r\right)\delta + t^{-2}E_\varphi u . \end{aligned} \tag{1.66}$$

Using the fact that $\varphi(0) = 1$ it now follows that

$$\begin{aligned} D(t)E_\varphi u &= 2\pi t^{-2}\left(\lim_{\varepsilon\downarrow 0}\int_\varepsilon^{t\varepsilon}\frac{1}{r}(1+\mathcal{O}(r))\,\mathrm{d}r\right)\delta + t^{-2}E_\varphi u \\ &= 2\pi t^{-2}\left(\lim_{\varepsilon\downarrow 0}\log(t\varepsilon) - \log(\varepsilon)\right)\delta + t^{-2}E_\varphi u \\ &= t^{-2}E_\varphi u + 2\pi t^{-2}\log(t)\delta\ . \end{aligned} \tag{1.67}$$

Thus, by Proposition 1.18 part (e) it follows that

$$D(t)\mathcal{F}[E_\varphi u] = \mathcal{F}[E_\varphi u] + \log t\ . \tag{1.68}$$

Because $E_\varphi u$ is rotationally invariant, it follows by Proposition 1.18 part (f) that $\mathcal{F}[E_\varphi u]$ is also rotationally invariant, hence there is some constant B, depending on the choice of φ, such that

$$\mathcal{F}[E_\varphi u](\xi) = \log|\xi| + B\ . \tag{1.69}$$

Of course a “good” choice of φ would be a φ such that $B = 0$. We set this distribution E_φ to be the *finite part* distribution $\mathbf{PF}r^{-2}$. In this case

$$\mathcal{F}[\mathbf{PF}r^{-2}](\xi) = \log|\xi|\ . \tag{1.70}$$

This whole construction is the dimension two case of Theorem 1.3.

For the higher dimensional cases note that both r^{-2} and r^{2-n} are locally absolutely integrable in $\mathbb{R}^n$, hence they both naturally define an element of $\mathcal{S}'(\mathbb{R}^n)$. Then by Proposition 1.18 part (i), it follows, for any Schwartz function v, that the identity $\langle v, r^{-2}\rangle = c\,\langle \hat{v}, r^{2-n}\rangle$ holds. In particular if we set $v = e^{-|x|^2/2}$, then $\hat{v} = e^{-|\xi|^2/2}$, so that

$$\int_{\mathbb{R}^n}\frac{e^{-|x|^2/2}}{|x|^2}\,\mathrm{d}x = c\int_{\mathbb{R}^n}\frac{e^{-|x|^2/2}}{|x|^{n-2}}\,\mathrm{d}x \tag{1.71}$$

The left-hand side can be simplified by

$$\begin{aligned} \left\langle v, r^{-2}\right\rangle &= A_{n-1}\int_0^\infty r^{n-3}e^{-r^2/2} = 2^{(n-4)/2}A_{n-1}\int_0^\infty s^{(n-4)/2}e^{-s}\,\mathrm{d}s \\ &= 2^{(n-4)/2}A_{n-1}\Gamma\left(\frac{n}{2}-1\right) \end{aligned} \tag{1.72}$$

whereas the right-hand side can be simplified by

$$\left\langle \hat{v}, r^{2-n}\right\rangle = A_{n-1}\int_0^\infty r^{2-n}r^{n-1}e^{-r^2/2} = A_{n-1}\int_0^\infty e^{-s}\,\mathrm{d}s = A_{n-1}\ . \tag{1.73}$$

Hence, we find an expression for c given by

$$c = 2^{(n-4)/2}\Gamma\left(\frac{n}{2}-1\right)\ . \tag{1.74}$$

Consequently, we have proven the second part of Theorem 1.3. ■

More generally, using a similar construction, we can prove the following proposition.

Proposition 1.25. For $m = -n - j$ with $j = 0, 1, 2, \ldots$ and for any $u_m \in \mathcal{S}'(\mathbb{R}^n)$ of the form of

$$u_m(x) = |x|^m \omega(x/|x|) \quad x \neq 0 \ , \tag{1.75}$$

with $\omega \in C^\infty(S^{n-1})$, there is an extension Eu_m such that $\mathcal{F}[Er^m]$ exists and is given by

$$\mathcal{F}[Eu_m](\xi) = w_j(\xi) + p_j(\xi) \log |\xi| \tag{1.76}$$

where $w_j \in \mathcal{H}_j^\#(\mathbb{R}^n)$ and $p_j(\xi)$ is a homogeneous polynomial of degree j.

Proof. First take any radial Schwartz function $\varphi(x)$ for which it holds that $\varphi(0) = 1$ and $1 - \varphi$ vanishes with order at least j. Then define the tempered distribution $E_{\varphi,j}u_m$ by the prescription

$$\langle v, E_{\varphi,j}u_m \rangle = \int_{\mathbb{R}^n} u_m(x) \left[v(x) - \sum_{|\alpha| \leq j} \frac{v^{(\alpha)}(0)}{\alpha!} x^\alpha \varphi(x) \right] \mathrm{d}x \ . \tag{1.77}$$

Then, once again, $E_{\varphi,j}u_m$ agrees with $u_m(x)$ on $\mathbb{R}^n \setminus 0$, because, again the prescription $E_{\varphi,j}u_m$ can be represented informally as

$$E_{\varphi,j}u_m = u_m(x) \left[1 - \sum_{|\alpha| \leq j} \frac{\delta^{(\alpha)}(x)}{\alpha!} x^\alpha \varphi(x) \right] \ . \tag{1.78}$$

Furthermore, the dependence on φ is then given by

$$\begin{aligned} &\langle v, E_{\varphi,j} - E_{\psi,j} \rangle \\ &\qquad = \sum_{|\alpha| \leq j} \frac{1}{\alpha!} \left(\int_{\mathbb{R}^n} u_m(x)(\psi(x) - \varphi(x)) x^\alpha \, \mathrm{d}x \right) \left\langle v, \delta^{(\alpha)} \right\rangle \ . \end{aligned} \tag{1.79}$$

As before, we see that

$$D(t)E_{\varphi,j}u_m = t^m E_{D(t)\varphi,j}u_m \ , \tag{1.80}$$

such that

$$\begin{aligned} D(t)E_{\varphi,j}u_m &= t^m \left[E_{D(t)\varphi,j}u_m - E_{\varphi,j}u_m + E_{\varphi,j}u_m \right] \\ &= t^m E_{\varphi,j}u_m + t^m \sum_{|\alpha| < j} \gamma_\alpha (t^{j-|\alpha|} - 1)\delta^{(\alpha)} + t^m \log t \sum_{|\alpha| = j} \gamma_\alpha \delta^{(\alpha)} \ . \end{aligned} \tag{1.81}$$

This is because by careful manipulations it holds that

$$
\begin{aligned}
&\sum_{|\alpha|\le j}\frac{1}{\alpha!}\left[\int_{\mathbb{R}^n} x^\alpha |x|^m \omega(x/|x|)(\varphi - D(t)\varphi)\,\mathrm{d}x\right] \\
&= \lim_{\varepsilon\downarrow 0}\sum_{|\alpha|\le j}\frac{1}{\alpha!}\left[\int_{\mathbb{R}^n\setminus B_\varepsilon} x^\alpha |x|^m\omega(x/|x|)\varphi(x)\,\mathrm{d}x\right. \\
&\qquad \left. - \int_{\mathbb{R}^n\setminus B_\varepsilon} x^\alpha |x|^m \omega(x/|x|)\varphi(tx)\,\mathrm{d}x\right] \\
&= \lim_{\varepsilon\downarrow 0}\sum_{|\alpha|<j}\frac{[1-t^{-n-|\alpha|-m}]}{\alpha!}\int_{\mathbb{R}^n\setminus B_\varepsilon} x^\alpha|x|^m\omega(x/|x|)\varphi(x)\,\mathrm{d}x \\
&\qquad + \sum_{|\alpha|=j}\frac{1}{\alpha!}\int_{A(\varepsilon,t\varepsilon)} x^\alpha |x|^m\omega(x/|x|)\varphi(x)\,\mathrm{d}x \\
&= \sum_{|\alpha|<j}\gamma_\alpha(t^{j-|\alpha|}-1) + \sum_{|\alpha|=j}\gamma_\alpha \log t\ .
\end{aligned}
\tag{1.82}
$$

Here $A(\varepsilon, t\varepsilon)$ denotes the spherical shell with inner radius $\min\{\varepsilon, t\varepsilon\}$ and outer radius $\max\{\varepsilon, t\varepsilon\}$. Then, if we set Eu_m to be the extension given by

$$
Eu_m = E_{\varphi,j}u_m - \sum_{|\alpha|<j}\gamma_\alpha\delta^{(\alpha)}\ , \tag{1.83}
$$

we see that Eu_m agrees with u_m on $\mathbb{R}^n\setminus 0$, and because $m = -n-j$ we also see that

$$
\begin{aligned}
D(t)Eu_m &= D(t)E_{\varphi,j}u_m - \sum_{|\alpha|<j}\gamma_\alpha t^{-n-|\alpha|}\delta^{(\alpha)} \\
&= t^m E_{\varphi,j}u_m + t^m\sum_{|\alpha|<j}\gamma_\alpha(t^{j-|\alpha|}-1)\delta^{(\alpha)} \\
&\quad + t^m\log t\sum_{|\alpha|=j}\gamma_\alpha\delta^{(\alpha)} - \sum_{|\alpha|<j}\gamma_\alpha t^{-n-|\alpha|}\delta^{(\alpha)} \\
&= t^m E_{\varphi,j}u_m - t^m\sum_{|\alpha|<j}\gamma_\alpha\delta^{(\alpha)} + t^m\log t\sum_{|\alpha|=j}\gamma_\alpha\delta^{(\alpha)} \\
&= t^m Eu_m + t^m\log t\sum_{|\alpha|=j}\gamma_\alpha\delta^{(\alpha)}\ .
\end{aligned}
\tag{1.84}
$$

Hence, for the Fourier transform of Eu_m it follows that

$$
D(t)\mathcal{F}[Eu_m] = t^j\mathcal{F}[Eu_m] + t^j\log t\sum_{|\alpha|=j}\gamma'_\alpha\xi^\alpha\ . \tag{1.85}
$$

Suppose now that for $|\xi| = 1$ it holds that $\mathcal{F}[Eu_m] = \varpi(\xi)$, then

$$
\mathcal{F}[Eu_m](t\xi) = t^j\varpi(\xi) + \log t\sum_{|\alpha|=j}\gamma'_\alpha(t\xi)^\alpha, \quad \text{for } |\xi| = 1\ , \tag{1.86}
$$

and hence

$$\mathcal{F}[Eu_m] = w_j(\xi) + p_j(\xi)\log|\xi| \tag{1.87}$$

where $w_j(\xi) \in \mathcal{H}_j^{\#}(\mathbb{R}^n)$ and $p_j(\xi)$ is a homogeneous polynomial of degree j. ■

Corollary 1.26. Suppose now further that u_m is of the form r^{-n-j} with j a non-negative integer, then there are constants C_1, C_2 such that $\mathcal{F}[Eu_m]$ is given by

$$\mathcal{F}[Eu_m] = C_1|\xi|^j + C_2|\xi|^j \log|\xi| \ . \tag{1.88}$$

Proof. The rotational invariance of u_m implies rotational invariance of Eu_m. Then, the Fourier transform $\mathcal{F}[Eu_m]$ is rotationally invariant. Equation (1.88) now follows, since the only rotationally invariant homogeneous distribution of degree j is $C_1|\xi|^j$ and since the only rotationally invariant homogeneous polynomial of degree j is $C_2|\xi|^j$. ■

The following calculations of Fourier transforms will be used in Chapter 3.

Proposition 1.27. The Fourier transform of $1/|x|^4$ on $\mathbb{R}^2$ is given by

$$\mathcal{F}\left[|x|^{-4}\right] = \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi|^2\log|\xi| \ . \tag{1.89}$$

Proof. Notice that from the previous corollary it is sufficient to find the constants C_1, C_2. Notice furthermore, that

$$C_2 = \lim_{\xi\to 0} \frac{\Delta_\xi \mathcal{F}[|x|^{-4}]}{2\log|\xi|} \ . \tag{1.90}$$

On the other hand by taking φ to be the radial function giving the $\mathbf{PF}r^{-2}$ extension of r^{-2}, we find that the distribution $|x|^2 \cdot E_\varphi |x|^{-4}$ is equal to the $\mathbf{PF}r^{-2}$. Indeed:

$$\begin{aligned} \left\langle v, |x|^2 E_\varphi |x|^{-4}\right\rangle &= \left\langle |x|^2 v, E_\varphi |x|^{-4}\right\rangle \\ &= \int_{\mathbb{R}^2} |x|^{-4}\left(|x|^2 v(x) - |x|^2 v(0)\varphi(x)\right) \mathrm{d}x \\ &= \int_{\mathbb{R}^2} |x|^{-2}\left(v(x) - v(0)\varphi(x)\right) \mathrm{d}x \\ &= \left\langle v, \mathbf{PF}r^{-2}\right\rangle \ , \end{aligned} \tag{1.91}$$

hence

$$C_2 = \lim_{\xi\to 0} \frac{\Delta_\xi \mathcal{F}[|x|^{-4}]}{2\log|\xi|} = \lim_{\xi\to 0} \frac{-\mathcal{F}[\mathbf{PF}r^{-2}]}{2\log|\xi|} = -\frac{1}{2} \ . \tag{1.92}$$

Similarly, we have that

$$C_1 + C_2 = \lim_{|\xi|\to 1} \frac{\Delta_\xi \mathcal{F}[|x|^{-4}]}{2} = \lim_{|\xi|\to 1} \frac{-\mathcal{F}[\mathbf{PF}r^{-2}]}{2} = 0 \tag{1.93}$$

which finishes the proof. ■

We will use these results in Chapter 3 to calculate the *Schwartz Kernel* of the leading terms of the *parametrix* of the Laplace-Beltrami operator on a two-dimensional space. The next results will allow us to do the same but in a three-dimensional space.

Proposition 1.28. The Fourier transform of $x_i x_j/|x|^6$ on $\mathbb{R}^3$ is a given by

$$\frac{\sqrt{2\pi}}{16}\left(\delta_{ij}|\xi| + \frac{\xi_i\xi_j}{|\xi|}\right) . \tag{1.94}$$

Proof. Corollary 1.26 gives that

$$\mathcal{F}[|x|^{-6}] = C_1|\xi|^3 + C_2|\xi|^3\log|\xi| \tag{1.95}$$

hence by Proposition 1.18 part (b) it follows that

$$\begin{aligned}\mathcal{F}\left[\frac{x_i^2}{|x|^6}\right] &= \partial^2_{\xi_i}\mathcal{F}\left[|x|^{-6}\right] \\ &= \frac{3(|\xi|^2 + |\xi_i|^2)(C_1 + C_2\log|\xi|) + C_2(|\xi|^2 + 4|\xi_i|^2)}{|\xi|} ,\end{aligned} \tag{1.96}$$

and that

$$\mathcal{F}\left[\frac{x_i x_j}{|\xi|^6}\right] = \partial_{\xi_i}\partial_{\xi_j}\mathcal{F}\left[|x|^{-6}\right] = \frac{\xi_i\xi_j\left(3C_1 + 4C_2 + 3C_2\log|\xi|\right)}{|\xi|} . \tag{1.97}$$

We calculate the constant C_2 first, and show that it is 0. Indeed,

$$\begin{aligned}12C_2 &= \lim_{\xi\to 0}\frac{\Delta_\xi\mathcal{F}[|x|^{-6}]}{|\xi|\log|\xi|} = \lim_{\xi\to 0}\frac{\mathcal{F}[-|x|^2\cdot|x|^{-6}]}{|\xi|\log|\xi|} \\ &= \lim_{\xi\to 0}\frac{-D_1|\xi| - D_2|\xi|\log|\xi|}{|\xi|\log|\xi|} = -D_2 ,\end{aligned} \tag{1.98}$$

and

$$\begin{aligned}D_2 &= \lim_{\xi\to 0}\frac{1}{2}\frac{|\xi|\Delta_\xi\mathcal{F}[|x|^{-4}]}{\log|\xi|} = \lim_{\xi\to 0}\frac{1}{2}\frac{|\xi|\mathcal{F}[-|x|^{-2}]}{\log|\xi|} \\ &= \lim_{\xi\to 0}\frac{1}{2}\frac{|\xi|}{\log|\xi|}\frac{-\sqrt{\pi/2}}{|\xi|} = 0 .\end{aligned} \tag{1.99}$$

Hence, $C_2 = 0$. Now we show that $C_1 = \sqrt{2\pi}/48$. Indeed,

$$12C_1 = \lim_{\xi\to 0}\frac{\Delta_\xi\mathcal{F}[|x|^{-6}]}{|\xi|} = \lim_{\xi\to 0}\frac{\mathcal{F}[-|x|^2\cdot|x|^{-6}]}{|\xi|} = -D_1 \tag{1.100}$$

and

$$D_1 = \lim_{\xi\to 0}\frac{1}{2}|\xi|\Delta_\xi\mathcal{F}[|x|^{-4}] = \lim_{\xi\to 0}\frac{1}{2}|\xi|\mathcal{F}[-|x|^{-2}] = -\frac{1}{4}\sqrt{2\pi} . \tag{1.101}$$

Thus, $C_1 = \sqrt{2\pi}/48$ and hence the lemma is proven. ∎

Corollary 1.29. The Fourier transform of $|x|^{-4}$ on $\mathbb{R}^3$ is given by

$$-\frac{1}{4}\sqrt{2\pi}|\xi| . \tag{1.102}$$

1.5 Partitions of unity

This section is dedicated to the creation of a partition of unity on $\mathbb{R}^n$. We show that such a partition of unity exists, and give some properties.

Lemma 1.30 (Partitions of unity). On $\mathbb{R}^n$ there is a partition of unity $\{\psi_k\}_{k=0}^{\infty}$ of compactly supported smooth functions, with

(a) $0 \le \psi_k(\xi) \le 1$,

(b) For each $\xi \in \mathbb{R}^n$ there are at least one and at most only three k such that $\psi_k(\xi) \neq 0$,

(c) $\sum_{k=0}^{\infty} \psi_k(\xi) = 1$ for all $\xi \in \mathbb{R}^n$,

(d) $\operatorname{supp} \psi_0 \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$,

(e) $\operatorname{supp} \psi_k \subseteq \{\xi \in \mathbb{R}^n : 2^{k-2} \le |\xi| \le 2^{k+1}\}$,

(f) For each multi-index α, there is some constant A_α that only depends on the multi-index α, such that

$$\sup_{\xi \in \mathbb{R}^n} |(\partial_\xi^\alpha \psi_k)(\xi)| \le A_\alpha 2^{-k|\alpha|} . \tag{1.103}$$

Proof. Take a compactly supported smooth function ϕ with the following properties

(a) $0 \le \phi(\xi) \le 1$

(b) $\phi(\xi) = 1$ for $1 \le |\xi| \le 2$

(c) $\phi(\xi) = 0$ whenever $|\xi| < \frac{1}{2}$ or $|\xi| > 4$

This function does indeed exist. Take $f(t) = e^{-1/t}$ for $t > 0$ and $f(t) = 0$ for $t \le 0$, then define

$$\tilde{\phi}(t) = \begin{cases} \frac{f(t-\frac{1}{2})}{f(t-\frac{1}{2})+f(1-t)} , & \text{for } t \le \frac{3}{2} \\ \frac{f(4-t)}{f(4-t)+f(t-2)} , & \text{for } t > \frac{3}{2} \end{cases} \tag{1.104}$$

Then this function satisfies all the properties above, as can be seen in Figure 1.1. It is a smooth function by construction, and has compact support, again by construction.

Now set $\phi(\xi) = \tilde{\phi}(|\xi|)$, $\phi_0(\xi) = \tilde{\phi}(|\xi| + 2)$ and $\phi_k(\xi) = \phi(\xi/2^{k-1})$, for $k \ge 1$ then

(a) For each $\xi \in \mathbb{R}^n$ there are at least one and at most only three k such that $\phi_k(\xi) \neq 0$

(b) $\operatorname{supp} \phi_0 \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$

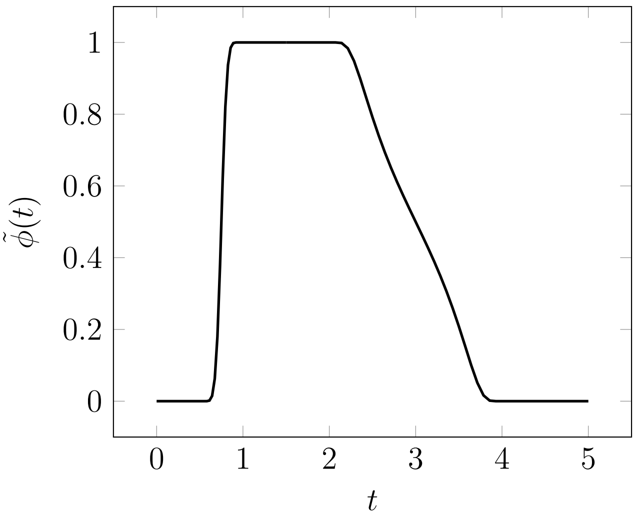

Figure 1.1: Cut-off functions

(c) $\operatorname{supp}\phi_k \subseteq \{\xi \in \mathbb{R}^n : 2^{k-2} \leq |\xi| \leq 2^{k+1}\}$

Set $\Phi(\xi) = \sum_{k=0}^{\infty} \phi_k(\xi)$, and let

$$\psi_k(\xi) = \frac{\phi_k(\xi)}{\Phi(\xi)} \tag{1.105}$$

then ψ_k satisfies properties (a) through (e) of the lemma. Now to prove part (f) of the lemma, calculate the derivative

$$\begin{aligned}(\partial^\alpha \psi_k)(\xi) &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \left(\partial^\beta \frac{1}{\Phi}(\xi)\right) \left(\partial^{\alpha-\beta}\phi_k(\xi)\right) \\ &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \left(\partial^\beta \left(\frac{1}{\Phi}(\xi)\right)\right) \left(\partial^{\alpha-\beta}\phi(\xi/2^{k-1})\right) \cdot 2^{-(k-1)|\alpha-\beta|} \ .\end{aligned} \tag{1.106}$$

For each β the partial derivative

$$\begin{aligned}&\partial^\beta \left(\frac{1}{\Phi}\right)(\xi) \\ &= \sum_{\beta^{(1)}+\beta^{(2)}+\cdots+\beta^{(l)}=\beta} C_{\beta^{(1)},\beta^{(2)},\ldots,\beta^{(l)}} \frac{\left(\partial^{\beta^{(1)}}\Phi\right)\cdots\left(\partial^{\beta^{(l)}}\Phi\right)}{\Phi^{l+1}} \ ,\end{aligned} \tag{1.107}$$

where the sum is taken over all possible multi-indices $\beta^{(1)}, \ldots, \beta^{(l)}$ which partition β, and where $C_{\beta^{(1)},\beta^{(2)},\ldots,\beta^{(l)}}$ is a constant. We now claim that for each multi-index γ there is a constant C_γ, such that

$$|(\partial^\gamma \Phi)(\xi)| \leq C_\gamma 2^{-k|\gamma|} \tag{1.108}$$

for all $\xi \in \operatorname{supp}(\psi_k)$. If we assume this claim to be true, then for all $\xi \in \operatorname{supp}(\psi_k)$

$$\begin{aligned}\left|\partial^\beta\left(\frac{1}{\Phi}\right)(\xi)\right| &\leq \sum_{\beta^{(1)}+\beta^{(2)}+\cdots+\beta^{(l)}=\beta} |C_{\beta^{(1)},\beta^{(2)},\ldots,\beta^{(l)}}| \frac{C_{\beta^{(1)}}\cdots C_{\beta^{(1)}}2^{-k|\beta|}}{|\Phi(\xi)|^{l+1}} \\ &\leq C'_\beta 2^{-k|\beta|} ,\end{aligned} \tag{1.109}$$

and hence

$$|(\partial^\alpha\psi_k)(\xi)| \leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}C'_\beta 2^{-k|\beta|}C_{\alpha,\beta}2^{-(k-1)|\alpha-\beta|} = A_\alpha 2^{-k|\alpha|} , \tag{1.110}$$

which proves part (f). To finish we need to prove the claim made in Equation (1.108). We take three cases: $k = 0$, $k = 1$ and $k \geq 2$.

First take $k = 0$, then $\xi \in \operatorname{supp}\psi_0$ implies $\xi \in \operatorname{supp}\phi_j$ for $j = 0, 1, 2$, and hence

$$(\partial^\gamma\Phi)(\xi) = (\partial^\gamma\phi_0)(\xi) + \sum_{j=1}^{2}(\partial^\gamma\phi)(\xi/2^{j-1})2^{-(j-1)|\gamma|} . \tag{1.111}$$

Thus, because all functions are compactly supported, any of their derivatives are compactly supported, which have a maximum such that

$$|(\partial^\gamma\Phi)(\xi)| \leq C_\gamma . \tag{1.112}$$

Similarly, if $k = 1$, then for all $\xi \in \operatorname{supp}(\psi_1)$ we have $\xi \in \operatorname{supp}(\phi_j)$ for $j = 0, 1, 2, 3$. Thus,

$$(\partial^\gamma\Phi)(\xi) = (\partial^\gamma\phi_0)(\xi) + \sum_{j=1}^{3}(\partial^\gamma\phi)(\xi/2^{j-1})2^{-(j-1)|\gamma|} , \tag{1.113}$$

such that

$$\begin{aligned}|(\partial^\gamma\Phi)(\xi)| &\leq C'_\gamma(2^{|\gamma|} + 1 + 2^{-|\gamma|} + 2^{-2|\gamma|}) \\ &= C'_\gamma(2^{2|\gamma|} + 2^{|\gamma|} + 1 + 2^{-|\gamma|})2^{-|\gamma|}\end{aligned} \tag{1.114}$$

for all $\xi \in \operatorname{supp}(\psi_1)$.

Now assume that $k \geq 2$, then for all $\xi \in \operatorname{supp}(\psi_k)$ we have $\xi \in \operatorname{supp}(\phi_j)$ for $j = k-2, k-1, k, k+1, k+2$, and it holds that

$$\Phi(\xi) = \sum_{j=k-2}^{k+2}\phi_j(\xi) \tag{1.115}$$

thus

$$(\partial\Phi)(\xi) = \sum_{j=k-2}^{k+2}(\partial^\gamma\phi)(\xi/2^{j-1})2^{-(j-1)|\gamma|} . \tag{1.116}$$

Therefore, there is a constant C''_γ such that

$$\begin{aligned}|(\partial^\gamma\Phi)(\xi)| &\leq C''_\gamma(2^{-(k-3)|\gamma|} + 2^{-(k-2)|\gamma|} + 2^{-(k-1)|\gamma|} + 2^{-k|\gamma|}2^{-(k-1)|\gamma|}) \\ &= C'_\gamma(2^{3|\gamma|} + 2^{2|\gamma|} + 2^{|\gamma|} + 1 + 2^{-|\gamma|})2^{-k|\gamma|} ,\end{aligned} \tag{1.117}$$

which finishes the proof of the claim and the lemma. ■

2 Pseudodifferential operators

In this chapter we look at pseudodifferential operators on flat Euclidean spaces and is material adapted and expanded upon from [Won14]. We define pseudodifferential operators, look at how to compose two pseudodifferential operators. In the second section we show that pseudodifferential operators are bounded linear operators on L^2. Furthermore, we show that pseudodifferential operators can be expressed as an integral operator using its Schwartz kernel. In the third section we look at elliptic pseudodifferential operators, and give their approximate inverse, the so called parametrix. Finally, in the last section we look at some applications of pseudodifferential operators on Sobolev spaces, and give an elliptic regularity result.

2.1 Symbols

In this section we give an introduction to pseudodifferential operators. In particular, we give the definition of pseudodifferential operators, and show how to compose two pseudodifferential operators.

Definition 2.1 (Symbol)**.** Let k be a real number. We say that a smooth function $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a symbol if for all multi-indices α, β there is a constant $C_{\alpha,\beta} \geq 0$, which depends only on the multi-indices α, β, such that

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \leq C_{\alpha,\beta}(1 + |\xi|)^{k-|\beta|} \tag{2.1}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Denote the set of symbols of order k as $\mathcal{S}^k$.

Recall that in Definitions 1.2 and 1.4 we defined the symbol of a partial differential operator. This definition generalises the symbol of a partial differential operator. We also define homogeneous symbols.

Definition 2.2. We say that a symbol $a(x,\xi) \in \mathcal{S}^k$ of order k is homogeneous of degree k, if there is some positive constant R such that

$$D_\xi(t)a(x,\xi) := a(x,t\xi) = t^k a(x,\xi) \tag{2.2}$$

outside some compact set $\{|\xi| \leq R\}$.

We will see later that homogeneous symbols have additional properties.

Using Definition 2.1 we are able to define a pseudodifferential operator.

Definition 2.3 (Pseudodifferential operators)**.** For a symbol a of order k define a linear operator $T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by

$$\begin{aligned}(T_a f)(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi)\, \mathrm{d}\xi \\ &= \mathcal{F}^{-1}\left[a(x,\xi)\hat{f}(\xi)\right] \ .\end{aligned} \tag{2.3}$$

We will see at a later stage that this operator extends uniquely to an operator $T_a : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Asymptotic expansions

The sum of two symbols a_0 and a_1 of orders k_0 and k_1 respectively, is a symbol of order $\max\{k_0, k_1\}$. The following definitions, lemmata and propositions show that even a series of symbols is a symbol under certain conditions.

Definition 2.4 (Asymptotic expansion)**.** Let $a(x,\xi) \in \mathcal{S}^k$ be a symbol, and let $k = k_0 > k_1 > k_2 > \ldots > k_j \to -\infty$ be a sequence of decreasing numbers, and a_{k_j} be symbols of order k_j such that

$$a - \sum_{j=0}^{N-1} a_j \in \mathcal{S}^{k_N} \tag{2.4}$$

for every integer $N > 0$, then $\sum_{j=0}^{\infty} a_j$ is an asymptotic expansion of a, and write

$$a \sim \sum_{j=0}^{\infty} a_j \ . \tag{2.5}$$

The following proposition shows the uniqueness of an asymptotic expansion.

Proposition 2.5. Suppose $a \in \mathcal{S}^k$ is a symbol with an asymptotic expansion of a given by

$$a \sim \sum_{j=0}^{\infty} a_j \tag{2.6}$$

where $a_j \in \mathcal{S}^{k_j}$. Suppose further that two symbols a, b have the same asymptotic expansion, then $a - b \in \cap_{k\in\mathbb{R}} \mathcal{S}^k$ and $T_a - T_b = T_{a-b}$ is *infinitely smoothing*, i.e. for every $f \in \mathcal{S}'(\mathbb{R}^n)$, the function $T_{a-b}f \in C^\infty(\mathbb{R}^n)$ is smooth.

The following lemma shows that symbols of lower order can be included in higher order symbol classes.

Lemma 2.6. Let $a \in \mathcal{S}^k$ be a symbol of order k, then $a \in \mathcal{S}^l$ for any $l > k$.

Proof. Notice that since $a \in \mathcal{S}^k$

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{k-|\beta|} \leq C_{\alpha,\beta}(1+|\xi|)^{l-|\beta|} \ , \tag{2.7}$$

thus $a \in \mathcal{S}^l$ also. ■

Proof of Proposition 2.5. Notice that for every positive integer

$$a - b = \left(a - \sum_{j=0}^{N-1} a_j\right) - \left(b - \sum_{j=0}^{N-1} a_j\right) \in \mathcal{S}^{k_N} \tag{2.8}$$

then as $N \to \infty$, by Lemma 2.6, it follows that

$$c = a - b \in \lim_{N\to\infty} \bigcap_{k \geq k_N} \mathcal{S}^k = \bigcap_{k\in\mathbb{R}} \mathcal{S}^k \ . \tag{2.9}$$

The second statement we will defer to later at this stage, since we first need to define the pseudodifferential operator on tempered distributions. We will still call symbols $c \in \bigcap_{k\in\mathbb{R}} \mathcal{S}^k$ and their associated pseudodifferential operators T_c (infinitely) smoothing. ■

Corollary 2.7. Let $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol of order k, such that there is a positive constant $R > 0$, such that for all $|\xi| > R$ it holds that $a(x,\xi) = 0$ for all $x \in \mathbb{R}^n$, then $a \in \cap_{k\in\mathbb{R}}\mathcal{S}^k$ and T_a is infinitely smoothing.

Proof. Observe that there exists non-negative constants $C_{\alpha,\beta}$, which depends only on the multi-indices α, β, such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{k-|\beta|} \tag{2.10}$$

Observe that $(1+|\xi|)^{k-|\beta|}$ attains some maximum in the compact set $|\xi| \leq R$. By sufficiently changing the constant $C_{\alpha,\beta}$ to some $C'_{\alpha,\beta}$ notice that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C'_{\alpha,\beta}(1+|\xi|)^{k-m-|\beta|} \tag{2.11}$$

for any $m > 0$. Together with Lemma 2.6, this shows that $a \in \cap_{k\in\mathbb{R}}\mathcal{S}^k$, and by Proposition 2.5 T_a is infinitely smoothing. ■

We have now shown that if a has an asymptotic expansion, then a is unique up to a symbol of order $-\infty$. The following proposition states that if we have a sequence of decreasing symbols, then there is a symbol, which has the series as its asymptotic expansion.

Proposition 2.8. Let $k = k_0 > k_1 > k_2 > \ldots > k_j \to -\infty$ be a sequence of decreasing numbers and let a_j be symbols of order k_j respectively, then there is a symbol $a \in \mathcal{S}^k$ of order k such that

$$a \sim \sum_{j=0}^{\infty} a_j \ . \tag{2.12}$$

Proof. Take $0 \leq \phi(\xi) \leq 1 \in C_0^\infty(\mathbb{R}^n)$ a compactly supported smooth function such that $\phi = 1$ whenever $|\xi| \leq 1$ and $\phi = 0$ whenever $|\xi| \geq 2$, and define $\psi = 1 - \phi \in C^\infty(\mathbb{R}^n)$. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subseteq (0,1]$ be a decreasing sequence converging to 0, and set

$$a(x,\xi) = \sum_{j=0}^{\infty} \psi(\varepsilon_j \xi) a_j(x,\xi) \ . \tag{2.13}$$

This a priori infinite sum, has at any point (x_0, ξ_0) and any neighbourhood U around it only finitely many terms, since for every such point and neighbourhood, it is possible to find an integer N such that for every $j > N$

$$\psi(\varepsilon_j \xi) a_j(x,\xi) = 0 \tag{2.14}$$

for every $(x,\xi) \in U$. Hence, $a(x,\xi)$ from Equation (2.13) is a smooth function $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

To show that $a(x,\xi)$ is indeed a symbol, it is required to show that for any two multi-indices α, β there exists some constant $C_{\alpha,\beta}$, which depends only on the multi-indices α and β, such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{k-|\beta|} \ . \tag{2.15}$$

By the chain rule

$$|D_\xi^\alpha \psi(\varepsilon\xi)| \leq A_\alpha \varepsilon^{|\alpha|} \qquad \text{where } A_\alpha = \sup_{\xi\in\mathbb{R}^n} |D_\xi^\alpha \psi(\xi)|. \tag{2.16}$$

Because $\operatorname{supp}\psi(\varepsilon\xi) \subseteq \{\xi \in \mathbb{R}^n : 2/\varepsilon \leq |\xi| \leq 1/\varepsilon\}$, it follows that $D_\xi^\alpha(\psi(\varepsilon\xi))$ is non-zero whenever $\varepsilon \leq 2/|\xi| \leq 4/(1+|\xi|)$. Then $|D_\xi^\alpha \psi(\varepsilon\xi)| \leq A'_\alpha(1+|\xi|)^{-|\alpha|}$ and

$$\begin{aligned}
&|D_x^\alpha D_\xi^\beta \psi(\varepsilon\xi) a_j(x,\xi)| \\
&\quad = \left| \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} A'_{\beta-\gamma}(1+|\xi|)^{|\gamma|-|\beta|} C_{\alpha,\beta-\gamma}(1+|\xi|)^{k_j-|\gamma|} \right| \\
&\quad \leq \tilde{C}_{\alpha,\beta}(1+|\xi|)^{-1}(1+|\xi|)^{k_j+1-|\beta|} \leq 4\varepsilon \tilde{C}_{\alpha,\beta,j}(1+|\xi|)^{k_j+1-|\beta|} \ .
\end{aligned} \tag{2.17}$$

Choose ε_j decreasing so, that $4\varepsilon_j \tilde{C}_{\alpha,\beta} < 2^{-j}$ for each j. Then set J so that for all $j > J$ it holds that $k_j + 1 \leq k_0$ then

$$\begin{aligned}
&|D_x^\alpha D_\xi^\beta a(x,\xi)| \\
&\quad \leq \sum_{j=0}^{J} \left| D_x^\alpha D_\xi^\beta \psi(\varepsilon_j\xi) a_j(x,\xi) \right| + \sum_{j=J+1}^{\infty} \left| D_x^\alpha D_\xi^\beta \psi(\varepsilon_j\xi) a_j(x,\xi) \right| \\
&\quad \leq \sum_{j=0}^{J} C_{\alpha,\beta,j}(1+|\xi|)^{k_0-|\beta|} + \sum_{j=J+1}^{\infty} 4\varepsilon_j \tilde{C}_{\alpha,\beta,j}(1+|\xi|)^{k_j+1-|\beta|} \\
&\quad < C'_{\alpha,\beta}(1+|\xi|)^{k_0-|\beta|} + 2^{-J}(1+|\xi|)^{k_0-|\beta|} \leq C_{\alpha,\beta}(1+|\xi|)^{k_0-|\beta|}
\end{aligned} \tag{2.18}$$

since the first sum is a finite sum of symbols of order k_0 by Lemma 2.6 and since $0 \leq \psi \leq 1$. The conclusion of this calculation is that $a(x,\xi)$ is a symbol of order $k = k_0$.

Now

$$a(x,\xi) - \sum_{j=0}^{N-1} a_j(x,\xi) = -\sum_{j=0}^{N-1} \phi(\varepsilon_j\xi)a_j(x,\xi) + \sum_{j=N}^{\infty} \psi(\varepsilon_j\xi)a_j(x,\xi) \tag{2.19}$$

is a symbol of order k_N, since $\sum_{j=0}^{N-1} \phi(\varepsilon_j\xi)a_j(x,\xi)$ is compactly supported in ξ, and hence an element of the intersection $\cap_{k\in\mathbb{R}}\mathcal{S}^k$ by Corollary 2.7, and

$$\sum_{j=N}^{\infty} \psi(\varepsilon_j\xi)a_j(x,\xi) \in \mathcal{S}^{k_N} \tag{2.20}$$

is a symbol precisely of order k_N by identical calculations as above. ■

Composition of pseudodifferential operators

We use the theory from the previous subsection to show that the composition of two pseudodifferential operators once again yields a pseudodifferential operator.

Theorem 2.1 (Composition of pseudodifferential operators)**.** *Let a, b be symbols of order k_1 and k_2 respectively, and let T_a, T_b be the corresponding pseudodifferential operators. Then there is a symbol $c := a\sharp b$ which has an asymptotic expansion given by*

$$(a\sharp b)(x,\xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi)) \tag{2.21}$$

of order $k_1 + k_2$ such that $T_c = T_a \circ T_b$ is a pseudodifferential operator.

A small lemma precedes the proof of the theorem.

Lemma 2.9. Given a symbol $a \in \mathcal{S}^k$ of order k, and the partition of unity ψ_k as defined in Lemma 1.30 there are symbols $a_k(x,\xi) = \psi_k(\xi)a(x,\xi)$, such that

$$T_a = \sum_{k=0}^{\infty} T_{a_k} \tag{2.22}$$

Proof. By construction, the sum of the symbols $a_k(x,\xi)$ is $a(x,\xi)$. Furthermore, by positivity of the integrand, it follows that

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |a_k(x,\xi)| \cdot |\hat{f}(\xi)| \,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} |a_k(x,\xi)| \cdot |\hat{f}(\xi)| \,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |a(x,\xi)| \cdot |\hat{f}(\xi)| \,\mathrm{d}\xi < \infty \end{aligned} \tag{2.23}$$

because $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$, and $|a(x,\xi)| \le C_{0,0}(1+|\xi|)^k$. Hence, by Tonelli's and Fubini's theorema for exchanging integrals and series, it follows that

$$\begin{aligned} \sum_{k=0}^{\infty} T_{a_k} f(x) &= \sum_{k=0}^{\infty} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a_k(x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sum_{k=0}^{\infty} a_k(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi = (T_a f)(x)\ . \end{aligned} \tag{2.24}$$

Hence, it follows that $T_a = \sum T_{a_k}$. ■

Proof of Theorem 2.1. Look at the composition of operators

$$\begin{aligned} (T_{a_k} \circ T_b f)(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a_k(x,\xi) \left(T_b f\right)^{\wedge}(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a_k(x,\xi) \int_{\mathbb{R}^n} e^{-i\xi\cdot y} (T_b f)(y)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n} (T_b f)(y) \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_k(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}y \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (T_b f)(y) K_k(x, x-y)\,\mathrm{d}y\ , \end{aligned} \tag{2.25}$$

where the interchange of integration is justified by the support of $a_k(x,\xi)$ being compact in ξ, and where $K_k(x, x-y)$ is the Schwartz kernel of $a_k(x,\xi)$ defined by

$$K_k(x,z) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iz\cdot\xi} a_k(x,\xi)\,\mathrm{d}\xi\ . \tag{2.26}$$

Then

$$\begin{aligned} (T_{a_k} \circ T_b f)(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (T_b f)(y) K_k(x, x-y)\,\mathrm{d}y \\ &= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n} K_k(x, x-y) \int_{\mathbb{R}^n} e^{iy\cdot\eta} b(y,\eta) \hat{f}(\eta)\,\mathrm{d}\eta\,\mathrm{d}y \\ &= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n} e^{ix\cdot\eta} \int_{\mathbb{R}^n} K_k(x, x-y) e^{i(y-x)\cdot\eta} b(y,\eta)\,\mathrm{d}y\ \hat{f}(\eta)\,\mathrm{d}\eta \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\eta} c_k(x,\eta) \hat{f}(\eta)\,\mathrm{d}\eta \end{aligned} \tag{2.27}$$

where c_k is the symbol defined by

$$\begin{aligned} c_k(x,\eta) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} K_k(x, x-y) e^{i(y-x)\cdot\eta} b(y,\eta)\,\mathrm{d}y \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} K_k(x,z) e^{-iz\cdot\eta} b(x-z,\eta)\,\mathrm{d}z\ . \end{aligned} \tag{2.28}$$

Now $c(x,\xi) = a\sharp b = \sum c_k(x,\xi)$, but what is still required to show, is that c indeed is a symbol of order $k_1 + k_2$ and that c does have the asymptotic expansion as given in Equation (2.21).

After taking a Taylor expansion of $b(x - z, \eta)$ around (x, η), by Proposition 1.18 part (b), and by the Fourier inversion formula, Proposition 1.18 part (h), it follows that

$$\begin{aligned} c_k(x,\eta) &= \sum_{|\mu|<N_1} \frac{(\partial_x^\mu b(x,\eta))}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{(-z)^\mu}{\mu!} e^{-iz\cdot\eta} K_k(x,z)\,\mathrm{d}z \\ &\quad + \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot\eta} K_k(x,z) R_{N_1}(x,z,\eta)\,\mathrm{d}z \\ &= \sum_{|\mu|<N_1} \frac{(-i)^{|\mu|}}{\mu!} \frac{(\partial_x^\mu b(x,\eta))}{(2\pi)^{n/2}} \partial_\eta^\mu \int_{\mathbb{R}^n} e^{-iz\cdot\eta} K_k(x,z)\,\mathrm{d}z \\ &\quad + \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot\eta} K_k(x,z) R_{N_1}(x,z,\eta)\,\mathrm{d}z \\ &= \sum_{|\mu|<N_1} \frac{(-i)^{|\mu|}}{\mu!} (\partial_x^\mu b(x,\eta))(\partial_\eta^\mu a_k(x,\eta)) \\ &\quad + \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot\eta} K_k(x,z) R_{N_1}(x,z,\eta)\,\mathrm{d}z \ , \end{aligned} \tag{2.29}$$

where the remainder $R_{N_1}(x,z,\eta)$ can be given in integral form

$$R_{N_1}(x,z,\eta) = N_1 \sum_{|\mu|=N_1} \frac{(-z)^\mu}{\mu!} \int_0^1 (1-\theta)^{N_1-1} (\partial_x^\mu b(x-\theta z,\eta))\,\mathrm{d}\theta \ . \tag{2.30}$$

Thus for every integer N_1 it is possible to write

$$c(x,\xi) = \sum_{|\mu|<N_1} \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu a(x,\xi))(\partial_x^\mu b(x,\xi)) + \sum_{k=0}^{\infty} T_{N_1}^{(k)}(x,\xi) \tag{2.31}$$

with

$$T_{N_1}^{(k)}(x,\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot\xi} K_k(x,z) R_{N_1}(x,z,\xi)\,\mathrm{d}z \ . \tag{2.32}$$

We conclude that the sequence of decreasing order symbols

$$\left(\sum_{|\mu|=N} \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu a(x,\xi))(\partial_x^\mu b(x,\xi)) \right)_{N\in\mathbb{N}} \tag{2.33}$$

now provides an asymptotic expansion of $c(x,\xi)$ if the remainder terms are sufficiently decreasing, since for any integer N_1 larger than N it holds that

$$\begin{aligned} c(x,\xi) - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu a(x,\xi))(\partial_x^\mu b(x,\xi)) \\ = c(x,\xi) - \sum_{|\mu|<N_1} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu a(x,\xi))(\partial_x^\mu b(x,\xi)) \\ + \sum_{N\le|\mu|<N_1} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu a(x,\xi))(\partial_x^\mu b(x,\xi)) \\ = \sum_{k=0}^{\infty} T_{N_1}^{(k)}(x,\xi) + \sum_{N\le|\mu|<N_1} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu a(x,\xi))(\partial_x^\mu b(x,\xi)) \end{aligned} \tag{2.34}$$

lies in $\mathcal{S}^{k_1+k_2-N}$ if and only if $\sum_{k=0}^{\infty} T_{N_1}^{(k)}(x,\xi)$ lies in $\mathcal{S}^{k_1+k_2-N}$. The following lemma will establish this fact.

Lemma 2.10. For all non-negative integers M, there is a positive constant C_{α,β,M,N_1} such that

$$|D_x^\alpha D_\xi^\beta T_{N_1}^{(k)}| \le C_{\alpha,\beta,M,N_1}(1+|\xi|)^{k_2-2M} 2^{(k_1+2M-N_1)k} \tag{2.35}$$

for all $x,\xi \in \mathbb{R}^n$ and all $k = 0,1,2,\ldots$.

Assume this lemma for now. For all positive integers N, and all multi-indices α,β we can choose an integer M such that

$$(1+|\xi|)^{k_2-2M} \le (1+|\xi|)^{k_1+k_2-N-|\beta|} \ . \tag{2.36}$$

Now find an integer N_1 so large that

$$k_1 + 2M - N_1 < 0 \ . \tag{2.37}$$

Then by Lemma 2.10, and by the choices on M and N_1, it follows that

$$\begin{aligned} |D_x^\alpha D_\xi^\beta T_{N_1}^{(k)}| \\ = \left| D_x^\alpha D_\xi^\beta \left(c(x,\xi) - \sum_{|\mu|<N_1} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu a(x,\xi))(\partial_x^\mu b(x,\xi)) \right) \right| \\ \le \sum_{k=0}^{\infty} C_{\alpha,\beta,M,N_1}(1+|\xi|)^{k_1+k_2-N-|\beta|} 2^{(k_1+2M-N_1)k} \\ = C_{\alpha,\beta}(1+|\xi|)^{k_1+k_2-N-|\beta|} \ , \end{aligned} \tag{2.38}$$

for all $x,\xi \in \mathbb{R}^n$, and where

$$C_{\alpha,\beta} = C_{\alpha,\beta,M,N_1} \sum_{k=0}^{\infty} 2^{(k_1+2M-N_1)k} \ . \tag{2.39}$$

Hence, Theorem 2.1 is proved once we establish Lemma 2.10. ■

The following two lemmata imply Lemma 2.10.

Lemma 2.11. For all multi-indices α, β and non-negative integers N, there is a constant A such that

$$\int_{\mathbb{R}^n} |z|^N |\partial_x^\beta \partial_z^\alpha K_k(x,z)| \,\mathrm{d}z \le A 2^{(k_1+|\alpha|-N)k} \ . \tag{2.40}$$

Proof. For this proof we use the inequality

$$|z|^{2N} = \left(\sum_{j=1}^n z_j^2\right)^N = \sum_{|\gamma|=N} \binom{N}{\gamma} z^{2\gamma} \le n^N \sum_{|\gamma|=N} |z^\gamma|^2 \tag{2.41}$$

for any $z \in \mathbb{R}^n$ and any multi-index γ, which partitions N. We now find that by Plancherel's theorem

$$\begin{aligned} \int_{\mathbb{R}^n} |z^\gamma (\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 \,\mathrm{d}z &= \int_{\mathbb{R}^n} |\partial_\xi^\gamma (\xi^\alpha \partial_x^\beta a_k)(x,\xi)|^2 \,\mathrm{d}\xi \\ &= \int_{W_k} \left| \sum_{\gamma' \le \gamma} \binom{\gamma}{\gamma'} \partial_\xi^{\gamma'} \left(\xi^\alpha \partial_x^\beta a(x,\xi)\right) \partial_\xi^{\gamma-\gamma'} \psi_k(\xi) \right|^2 \mathrm{d}\xi \end{aligned} \tag{2.42}$$

where W_k is the support of ψ_k which we constructed in Lemma 1.30. Furthermore, by part (f) of that lemma, and by the fact that $\xi^\alpha \partial_x^\beta a(x,\xi)$ is a symbol of order $k_1 + |\alpha|$ it follows that there are constants $C_{\alpha,\beta,\gamma'}, A_{\gamma,\gamma'}$ such that

$$\begin{aligned} &\int_{\mathbb{R}^n} |z^\gamma (\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 \,\mathrm{d}z \\ &\quad \le \int_{W_k} \left(\sum_{\gamma' \le \gamma} \binom{\gamma}{\gamma'} C_{\alpha,\beta,\gamma'} (1+|\xi|)^{k_1+|\alpha|-|\gamma'|} A_{\gamma,\gamma'} 2^{-k|\gamma-\gamma'|} \right)^2 \mathrm{d}\xi \\ &\quad \le \int_{W_k} \left(\sum_{\gamma' \le \gamma} \binom{\gamma}{\gamma'} C_{\alpha,\beta,\gamma'} 2^{(k+2)(k_1+|\alpha|-|\gamma'|)} A_{\gamma,\gamma'} 2^{-k|\gamma-\gamma'|} \right)^2 \mathrm{d}\xi \end{aligned} \tag{2.43}$$

because $(1 + |\xi|)$ is bounded on W_k. Hence, it is possible to find a constant $A_{\alpha,\beta,\gamma,k_1,n}$, which depends only on $\alpha, \beta, \gamma, k_1$ and n, such that

$$\int_{\mathbb{R}^n} |z^\gamma (\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 \,\mathrm{d}z \le A_{\alpha,\beta,\gamma,k_1,n} 2^{k(n+2k_1+2|\alpha|-2|\gamma|)} \ . \tag{2.44}$$

Therefore, by Equation (2.41), there is a constant $A_{\alpha,\beta,k_1,n}$, which depends only on α, β, k_1 and n, such that

$$\int_{\mathbb{R}^n} |z|^{2N} |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 \,\mathrm{d}z \le A_{\alpha,\beta,k_1,n} 2^{k(n+2k_1+2|\alpha|-2N)} \ . \tag{2.45}$$

Finally, to compute

$$\begin{aligned} &\int_{\mathbb{R}^n} |z|^N |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)| \,\mathrm{d}z = I_1 + I_2 = \\ &\int_{|z| \le 2^{-k}} |z|^N |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)| \,\mathrm{d}z + \int_{|z| > 2^{-k}} |z|^N |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)| \,\mathrm{d}z \ , \end{aligned} \tag{2.46}$$

we use the Cauchy-Schwartz inequality on both integrals. Therefore, there is a constant A_1 depending on α, β, k_1, n, N such that

$$\begin{aligned} I_1 &\leq \left(\int_{\mathbb{R}^n} |z|^{2N} |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 \,\mathrm{d}z\right)^{1/2} \left(\int_{|z|\leq 2^{-k}} \mathrm{d}z\right) \\ &\leq A_1 2^{k(n/2+k_1+|\alpha|-N)} 2^{-nk/2} = A_1 2^{(k_1+|\alpha|-N)k} \,. \end{aligned} \tag{2.47}$$

Similarly, there is a constant A_2 depending on α, β, k_1, n, N such that

$$\begin{aligned} I_2 &\leq \left(\int_{\mathbb{R}^n} |z|^{2N+2n} |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 \,\mathrm{d}z\right)^{1/2} \left(\int_{|z|>2^{-k}} |z|^{-2n} \,\mathrm{d}z\right) \\ &\leq A_2 2^{k(n/2+k_1+|\alpha|-N-n)} |S^{n-1}| n^{-1/2} 2^{nk/2} = A_3 2^{(k_1+|\alpha|-N)k} \,. \end{aligned} \tag{2.48}$$

Here $|S^{n-1}|$ is the surface area of the hypersphere S^{n-1} and A_3 a constant equal to $A_2|S^{n-1}|n^{1/2}$. This finishes the proof. ■

Lemma 2.12. For all multi-indices α, β, γ there is a constant $C_{\alpha,\beta,\gamma}$ such that

$$|(\partial_z^\gamma \partial_x^\alpha \partial_\xi^\beta R_{N_1})(x,z,\xi)| \leq C_{\alpha,\beta,\gamma} (1+|\xi|)^{k_2-|\beta|} \sum_{\gamma'\leq\gamma} |z|^{N_1-|\gamma'|} \,. \tag{2.49}$$

Proof. We calculate using the definition of R_{N_1} from Equation (2.30) its derivatives

$$\begin{aligned} &\partial_z^\gamma \partial_\xi^\beta \partial_x^\alpha R_{N_1}(x,z,\xi) \\ &\quad = N_1 \sum_{|\mu|=N_1} \sum_{\gamma'\leq\gamma} \binom{\gamma}{\gamma'} \frac{\partial_z^{\gamma'}(-z)^\mu}{\mu!} \int_0^1 (1-\theta)^{N_1-1} \\ &\qquad\qquad (\partial_\xi^\beta \partial_x^{\mu+\alpha+\gamma-\gamma'} b(x-\theta z,\eta))(-\theta)^{|\gamma-\gamma'|} \,\mathrm{d}\theta \,. \end{aligned} \tag{2.50}$$

Thus using the fact that b is a symbol of order k_2 it follows that there are constants $C_{\alpha,\beta,\gamma,\gamma'}, C_{\gamma'}$ and $C_{\alpha,\beta,\gamma}$ such that

$$\begin{aligned} |(\partial_z^\gamma \partial_x^\alpha \partial_\xi^\beta R_{N_1})(x,z,\xi)| &\leq N_1 \sum_{|\mu|=N_1} \sum_{\gamma'\leq\gamma} \binom{\gamma}{\gamma'} \frac{C_{\gamma'}|z|^{N_1-|\gamma'|}}{\mu!} \int_0^1 (1-\theta)^{N_1-1} \\ &\qquad (C_{\alpha,\beta,\gamma,\gamma'}(1+|\xi|)^{k_2-|\beta|})(-\theta)^{|\gamma-\gamma'|} \,\mathrm{d}\theta \\ &\leq C_{\alpha,\beta,\gamma}(1+|\xi|)^{k_2-|\beta|} \sum_{\gamma'\leq\gamma} |z|^{N_1-|\gamma'|} \,, \end{aligned} \tag{2.51}$$

with

$$C_{\alpha,\beta,\gamma} = N_1 \sum_{|\mu|=N_1} \sup_{\gamma'\leq\gamma} \left\{ \binom{\gamma}{\gamma'} C_{\alpha,\beta,\gamma,\gamma'} C_{\gamma'} \right\} \tag{2.52}$$

since the integral

$$\int_0^1 (1-\theta)^{N_1-1} (-\theta)^{|\gamma-\gamma'|} \,\mathrm{d}\theta \leq 1 \,, \tag{2.53}$$

which finishes the proof. ■

Proof of Lemma 2.10. For each multi-index we calculate

$$\xi^\gamma(\partial_x^\alpha \partial_\xi^\beta T_{N_1}^{(k)})(x,\xi) \\ = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iz\cdot\xi}(i)^{|\gamma|}(-i)^{|\beta|} z^\beta \partial_x^\alpha \partial_z^\gamma \Big[K_k(x,z) R_{N_1}(x,z,\xi) \Big] \, \mathrm{d}z \tag{2.54}$$

Hence by Leibniz formula we find that

$$|\xi^\gamma(\partial_x^\alpha \partial_\xi^\beta T_{N_1}^{(k)})(x,\xi)| \\ \leq (2\pi)^{-n/2} \sum_{\alpha' \leq \alpha} \sum_{\gamma' \leq \gamma} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \\ \int_{\mathbb{R}^n} |z|^\beta |(\partial_x^{\alpha'} \partial_z^{\gamma'} R_{N_1})(x,z,\xi)| \cdot |\partial_x^{\alpha-\alpha'} \partial_z^{\gamma-\gamma'} K_k(x,z)| \, \mathrm{d}z \ . \tag{2.55}$$

Now by Lemma 2.12 we find that

$$|\xi^\gamma(\partial_x^\alpha \partial_\xi^\beta T_{N_1}^{(k)})(x,\xi)| \\ \leq (2\pi)^{-n/2} \sum_{\alpha' \leq \alpha} \sum_{\gamma' \leq \gamma} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} C_{\alpha',0,\gamma'} (1+|\xi|)^{k_2} \\ \int_{\mathbb{R}^n} |z|^{|\beta|} \sum_{\gamma'' \leq \gamma'} |z|^{N_1 - |\gamma''|} \cdot |\partial_x^{\alpha-\alpha'} \partial_z^{\gamma-\gamma'} K_k(x,z)| \, \mathrm{d}z \\ = (2\pi)^{-n/2} \sum_{\alpha' \leq \alpha} \sum_{\gamma' \leq \gamma} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \sum_{\gamma'' \leq \gamma'} C_{\alpha',0,\gamma'} (1+|\xi|)^{k_2} \\ \int_{\mathbb{R}^n} |z|^{|\beta| + N_1 - |\gamma''|} \cdot |\partial_x^{\alpha-\alpha'} \partial_z^{\gamma-\gamma'} K_k(x,z)| \, \mathrm{d}z \ , \tag{2.56}$$

which by Lemma 2.11 can be reduced to

$$|\xi^\gamma(\partial_x^\alpha \partial_\xi^\beta T_{N_1}^{(k)})(x,\xi)| \\ \leq (2\pi)^{-n/2} \sum_{\alpha' \leq \alpha} \sum_{\gamma' \leq \gamma} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \sum_{\gamma'' \leq \gamma'} C_{\alpha',0,\gamma'} (1+|\xi|)^{k_2} \\ \cdot A_{N_1} 2^{(k_1 + |\gamma-\gamma'| - |\beta| - N_1 + |\gamma''|)k} \\ = C_{\alpha,\beta,\gamma,N_1} (1+|\xi|)^{k_2} \cdot 2^{(k_1 + |\gamma| - |\beta| - N_1)k} \\ \leq C_{\alpha,\beta,\gamma,N_1} (1+|\xi|)^{k_2} \cdot 2^{(k_1 + |\gamma| - N_1)k} \ . \tag{2.57}$$

Now because $(1+|\xi|)^{2M} \leq C_M(1+|\xi|^2)^M = \sum_{|\gamma| \leq 2M} c_\gamma \xi^\gamma$, it follows that

$$(1+|\xi|)^{2M} \cdot |(\partial_x^\alpha \partial_\xi^\beta T_{N_1}^{(k)})(x,\xi)| \leq C_{\alpha,\beta,M,N_1} (1+|\xi|)^{k_2} \cdot 2^{(k_1+2M-N_1)k} \tag{2.58}$$

for any positive integer M, which finishes the proof for this lemma. ■

Using a similar proving techniques as we just used it is possible to extend pseudo-differential operators to the tempered distributions.

The formal adjoint of pseudodifferential operators

The following theorem which we do not prove, but which uses similar techniques as the proof of Theorem 2.1 proves the existence and uniqueness, up to smoothing operators, of the formal adjoint of a pseudodifferential operator.

Theorem 2.2. *Let $a(x,\xi)$ be a symbol of order k, and let T_a be the corresponding pseudodifferential operator, then the function $b(x,\xi)$ given by*

$$b(x,\xi) = \sum_{\alpha} \frac{(-i)^{\alpha}}{\alpha!} (\partial_x^{\alpha} \partial_{\xi}^{\alpha} \overline{a})(x,\xi) \tag{2.59}$$

is a symbol of order k and T_b is a pseudodifferential operator. Furthermore, for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ it holds that

$$(T_a f, g) = \int_{\mathbb{R}^n} (T_a f)(x) \overline{g(x)} \, \mathrm{d}x = \int_{\mathbb{R}^n} f(x) \overline{(T_b g)(x)} \, \mathrm{d}x = (f, T_b g) \ . \tag{2.60}$$

A proof can be found in [Won14] section 9.

We will denote the formal adjoint of a pseudodifferential operator T by T^*. Using the formal adjoint we can extend pseudodifferential operators to tempered distributions given in the following definition.

Definition 2.13. A pseudodifferential operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to an operator $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ under the prescription

$$(Tu)(f) = u(\overline{T^* \overline{f}}) \ . \tag{2.61}$$

This definition is consistent, since if $u \in \mathcal{S}(\mathbb{R}^n)$ then

$$u(\overline{T^* \overline{f}}) = \int_{\mathbb{R}^n} u(x) \overline{T^* \overline{f(x)}} \, \mathrm{d}x = \int_{\mathbb{R}^n} (Tu)(x) \overline{\overline{f}}(x) \, \mathrm{d}x = (Tu)(f) \ . \tag{2.62}$$

We can now also give the proof for the second part of Proposition 2.5.

Proof of the second part of Proposition 2.5. If $c \in \mathcal{S}^{-\infty} = \bigcap_{k \in \mathbb{R}} \mathcal{S}^k$ it follows that for every fixed $x_0 \in \mathbb{R}^n$ the function $c(\xi) := c(x_0, \xi)$ is a Schwartz function. Now let $v \in \mathcal{S}'(\mathbb{R}^n)$ be the Fourier transform of $u \in \mathcal{S}'(\mathbb{R}^n)$, then $v(c(\xi) e^{i x_0 \cdot \xi})$ exists and is finite, and continuous. Hence, we may set $(Tu)(x) = v(c(x,\xi) e^{i x \cdot \xi})$. Furthermore, because

$$\begin{aligned} &\lim_{t \to 0} \frac{v\left(c(x_0,\xi) e^{i x_0 \cdot \xi} - c(x_0 + t e_i, \xi) e^{i (x_0 + t e_i) \cdot \xi}\right)}{t} \\ &\qquad = v\left(\lim_{t \to 0} \xi_i c(x_0,\xi) + \partial_{x_i} c(x_0,\xi) + \mathcal{O}(t) \right) < \infty \ , \end{aligned} \tag{2.63}$$

it follows that any first partial derivatives exists. Because $\xi_i c(x_0,\xi)$ and $\partial_{x_i} c(x_0,\xi)$ are also Schwartz functions it now follows, by induction, that $T_c u$ is a smooth function in the classical sense. ■

Using the density of the Schwartz functions in the tempered distributions, Theorem 1.1, we are also able to extend pseudodifferential operators. The extensions agree on all tempered distributions.

In the next section we will delve deeper into Schwartz kernels, and use it to prove that pseudodifferential operators are bounded linear operators on $L^p(\mathbb{R}^n)$.

2.2 Schwartz kernels and L^p boundedness

Theorem 2.3 (Pseudodifferential operators are bounded linear operators)**.** *Let $a \in \mathcal{S}^0$ be a symbol, then $T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to a bounded linear operator $T_a : L^p \to L^p$ for all $p \in (1, \infty)$.*

The extension of the pseudodifferential operator to a bounded linear operator on L^p of course coincides with the definition of a pseudodifferential operator on tempered distributions. We first need some preparations for the proof.

Theorem 2.4. *Let $a(x, \xi) \in \mathcal{S}^0$ be a symbol of order* 0*. Set*

$$K(x, z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iz\cdot\xi} a(x, \xi) \, \mathrm{d}\xi \tag{2.64}$$

in a distributional sense. Then

(a) For each fixed $x \in \mathbb{R}^n$, $z \mapsto K(x, z)$ is a function defined on $\mathbb{R}^n \setminus \{0\}$,

(b) For each sufficiently large integer N, there is a positive constant C_N such that $|K(x, z)| \leq C_N |z|^{-N}$, whenever $z \neq 0$,

(c) For each fixed $x \in \mathbb{R}^n$, and each fixed $\varphi \in \mathcal{S}(\mathbb{R}^n)$ which vanishes on a neighbourhood around x, it holds that

$$(T_a\varphi)(x) = \int_{\mathbb{R}^n} K(x, x - z)\varphi(z) \, \mathrm{d}z \ . \tag{2.65}$$

Proof. Calculate for any multi-index α

$$(iz)^\alpha K(x, z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iz\cdot\xi} \partial_\xi^\alpha a(x, \xi) \, \mathrm{d}\xi \ , \tag{2.66}$$

such that for $N = |\alpha|$ sufficiently large we have that

$$\begin{aligned} |z|^N |K(x, z)| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\partial_\xi^\alpha a(x, \xi)| \, \mathrm{d}\xi \\ &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |(1 + |\xi|)^{-N}| \, \mathrm{d}\xi \end{aligned} \tag{2.67}$$

exists classically on $\mathbb{R}^n \setminus \{0\}$. This proves parts (a) and (b).

To prove part (c), define for each x the tempered distribution u_x by the prescription

$$u_x(\varphi) = \int_{\mathbb{R}^n} a(x,\xi)\varphi(\xi)\,\mathrm{d}\xi \qquad \varphi \in \mathcal{S}(\mathbb{R}^n)\ . \tag{2.68}$$

Then for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that φ vanishes on a neighbourhood around x, it follows that

$$\begin{aligned}(T_a\varphi)(x) &= (2\pi)^{-n/2} u_x\left(e^{ix\cdot\xi}\hat{\varphi}(\xi)\right)\\ &= (2\pi)^{-n/2} u_x\left((\varphi(z+x))^\wedge\right)\\ &= (2\pi)^{-n/2} \hat{u}_x\left(\varphi(z+x)\right)\end{aligned} \tag{2.69}$$

by part (d) of Proposition 1.18 and the definition of the Fourier transform of a tempered distribution. Now by part (a) of this theorem, it follows that

$$\begin{aligned}(T_a\varphi)(x) &= \int_{\mathbb{R}^n} K(x,-z)\varphi(z+x)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^n} K(x,x-z)\varphi(z)\,\mathrm{d}z\end{aligned} \tag{2.70}$$

for all φ vanishing in a neighbourhood around the origin. ■

The following theorem will show that pseudodifferential operators always have Schwartz kernels. The proof is omitted, but can be found in [FJ98].

Theorem 2.5 (Existence of the Schwartz kernel)**.** *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open sets. A linear map $\mu : C_0^\infty(Y) \to \mathcal{D}'(X)$ is sequentially continuous if and only if it is generated by a Schwartz kernel $k \in \mathcal{D}'(X \times Y)$,*

$$\langle \mu\psi, \phi\rangle = \langle k, \phi\otimes\psi\rangle \quad \phi \in C_0^\infty(X), \quad \psi \in C_0^\infty(Y)\ . \tag{2.71}$$

Moreover, k is uniquely determined by μ.

The consequence is the following.

Corollary 2.14. Since the map $T_a : C_0^\infty(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) : f \mapsto T_af$ is sequentially continuous, it follows that T_a is generated by a Schwartz kernel $K(x,y)$ which lies in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$.

In other words any pseudodifferential operator T_a can be written as the integral operator

$$(T_a\varphi)(x) = \int_{\mathbb{R}^n} K(x,x-z)\varphi(z)\,\mathrm{d}z \tag{2.72}$$

with $y \mapsto K(x,y)$ a function on $\mathbb{R}^n \setminus \{0\}$.

The Schwartz kernel allows us to represent a pseudodifferential operator as an integral operator. Using this integral operator we are able to give the singular support from Definition 1.20 of T_af given the singular support of f.

Lemma 2.15. For any pseudodifferential operator T_a the singular support of $T_a u$ is contained in the singular support of u.

Proof. Let $x_0 \notin \operatorname{sing\,supp} u$, and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi u \in C_0^\infty(\mathbb{R}^n)$, and $\varphi(x_0) = 1$. Now choose $\psi \in C_0^\infty(\mathbb{R}^n)$ a compactly supported smooth function $0 \leq \psi \leq 1$ such that $\psi \equiv 1$ on $\operatorname{supp} \varphi$ and $\psi u \in C^\infty(\mathbb{R}^n)$. Then, since we can write the pseudodifferential operator applied to u at x_0 as an integral operator, it follows that

$$\begin{aligned}(\varphi T_a u)(x_0) &= \int_{\mathbb{R}^n} \varphi(x_0) K(x_0, x_0 - z) u(z)\,\mathrm{d}z \\ &= \int_{\mathbb{R}^n} \varphi(x_0)\psi(z) K(x_0, x_0 - z) u(z)\,\mathrm{d}z \\ &\qquad + \int_{\mathbb{R}^n} \varphi(x_0)(1 - \psi(z)) K(x_0, x_0 - z) u(z)\,\mathrm{d}z\ , \\ &= (\varphi T_a(\psi u))(x_0) + \int_{\mathbb{R}^n} \varphi(x_0)(1 - \psi(z)) K(x_0, x_0 - z) u(z)\,\mathrm{d}z\end{aligned} \tag{2.73}$$

is smooth, since the first term is a pseudodifferential operator applied to a Schwartz function, and since the integrand in the second term is zero on a neighbourhood around the diagonal. Hence, $x_0 \notin \operatorname{sing\,supp} T_a u$, and therefore $\operatorname{sing\,supp} T_a u \subseteq \operatorname{sing\,supp} u$. ■

In general, we are able to see the Schwartz kernel of a pseudodifferential operator as the inverse Fourier transform of the symbol $a(x, \xi)$

$$K(x, z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iz\cdot\xi} a(x, \xi)\,\mathrm{d}\xi\ . \tag{2.74}$$

Hence, using Section 1.4 we immediately find the following proposition.

Proposition 2.16. Suppose that $a(x, \xi)$ is a homogeneous symbol of order k, then

$$D_z(t) K(x, z) := K(x, tz) = t^{-n-k} K(x, z)\ . \tag{2.75}$$

In Chapter 3 we will compute the Schwartz kernels of the parametrix of the Laplace-Beltrami operator. We will create the parametrix in a way such that the symbols in the asymptotic expansion of this parametrix are homogeneous. In the next section we define this parametrix. But now we move back to the proof of Theorem 2.3.

Proof of Theorem 2.3. It is sufficient to show that for any $a \in \mathcal{S}^0$, the pseudodifferential operator $T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bounded linear operator on $\mathcal{S}(\mathbb{R}^n)$ in the L^p-norm, because by the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, the pseudodifferential operator will extend to a bounded linear operator on L^p. We give the proof for $p = 2$, but with some slight modifications the proof also holds for any $p \in (1, \infty)$.

Now for the boundedness proof, let $\mathbb{Z}^n$ be the set of all n-tuples, with integer coefficients. Let $m \in \mathbb{Z}^n$ be vector, and let Q_m be a (hyper)-cube with centre m and edges of length one, parallel to the coordinate axes. Let $\eta(x) \in C_0^\infty(\mathbb{R}^n)$ be a compactly supported smooth function, such that $\eta(x) = 1$ for each $x \in Q_0$. Define $a_m(x,\xi)$ by

$$a_m(x,\xi) = \eta(x-m)a(x,\xi) \ , \tag{2.76}$$

then since $a_m(x,\xi)$ has compact support in x it follows that

$$\begin{aligned}(T_{a_m}\varphi)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}a_m(x,\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\left(\int_{\mathbb{R}^n} e^{ix\cdot\lambda}\hat{a}_m(\lambda,\xi)\,\mathrm{d}\lambda\right)\hat{\varphi}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\lambda}\left(\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{a}_m(\lambda,\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\lambda\end{aligned} \tag{2.77}$$

with $\hat{a}_m(\lambda,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\lambda\cdot x}a_m(x,\xi)\,\mathrm{d}x$, by the Fourier inversion formula. Set $T_\lambda : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ to be the operator defined by

$$(T_\lambda\varphi)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{a}_m(\lambda,\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi \ . \tag{2.78}$$

We will show later that for any integer N there is a positive constant C_N such that $|\hat{a}_m(\lambda,\xi)| \le C_N(1+|\lambda|)^{-N}$. Assuming this, it holds that

$$\|T_\lambda\varphi\|_2 = \|\hat{a}_m(\lambda,\cdot)\hat{\varphi}\|_2 \le (1+|\lambda|)^{-N}\|\hat{\varphi}\|_2 = (1+|\lambda|)^{-N}\|\varphi\|_2 \ , \tag{2.79}$$

by the Plancherel-Parseval identity, Proposition 1.18 part (i). Using this and by Minkowski's integral inequality

$$\begin{aligned}\|T_{a_m}\varphi\|_2 &= (2\pi)^{-n/2}\left\{\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{ix\cdot\lambda}(T_\lambda\varphi)(x)\,\mathrm{d}\lambda\Big|^2\mathrm{d}x\right\}^{1/2} \\ &\le (2\pi)^{-n/2}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|(T_\lambda\varphi)(x)\Big|^2\mathrm{d}x\right\}^{1/2}\mathrm{d}\lambda \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\|T_\lambda\varphi\|_2\,\mathrm{d}\lambda \\ &\le C_N(2\pi)^{-n/2}\|\varphi\|_2\int_{\mathbb{R}^n}(1+|\lambda|)^{-N}\,\mathrm{d}\lambda \le D_N\|\varphi\|_2 \ ,\end{aligned} \tag{2.80}$$

for sufficiently large N. Furthermore, it follows obviously that

$$\int_{Q_m}|(T_a\varphi)(x)|^2\,\mathrm{d}x \le \int_{\mathbb{R}^n}|(T_{a_m}\varphi)(x)|^2\,\mathrm{d}x \le D_N^2\|\varphi\|_2^2 \ . \tag{2.81}$$

We want to give a bound on the sum over all $m \in \mathbb{Z}^n$ of the integral above. Let Q_m^{**} be the cube with centre m and edges of length 2 parallel to the coordinate axes, and let Q_m^* be a cube with centre m and edges parallel to the coordinate axes such that $Q_m \subseteq Q_m^* \subseteq Q_m^{**}$, and such that there is a positive δ such that $|x-z| \geq \delta$ for all $x \in Q_m$ and all $z \in \mathbb{R}^n \setminus Q_m^*$.

Take a compactly supported smooth function $0 \leq \psi \leq 1$ with $\operatorname{supp} \psi \subseteq Q_m^{**}$ and $\psi(x) = 1$ on a neighbourhood of Q_m^*. Write $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 = \psi\varphi$ and $\varphi_2 = (1-\psi)\varphi$. Then clearly $T_a\varphi = T_a\varphi_1 + T_a\varphi_2$. Write

$$I_m = \int_{Q_m} |(T_a\varphi)(x)|^2 \,\mathrm{d}x \tag{2.82}$$

and

$$J_m = \int_{Q_m} |(T_a\varphi_2)(x)|^2 \,\mathrm{d}x \tag{2.83}$$

Then by the previous calculations, and by the convexity of $f(x) = |x|^p$ it follows that

$$\begin{aligned} I_m &= \int_{Q_m} |(T_a\varphi_1)(x) + (T_a\varphi_2)(x)|^2 \,\mathrm{d}x \\ &\leq 2\int_{Q_m} |T_a\varphi_1|^2 + 2J_m \\ &\leq 2D_N^2 \|\varphi_1\|_2^2 + 2J_m \ . \end{aligned} \tag{2.84}$$

We will now use Theorem 2.4 to give a bound on J_m. In particular for $x \in Q_m$ there is a constant C_{2N} such that

$$\begin{aligned} |(T_a\varphi_2)(x)| &= (2\pi)^{-n/2} \Big| \int_{\mathbb{R}^n} K(x, x-z)\varphi_2(z) \,\mathrm{d}z \Big| \\ &= (2\pi)^{-n/2} \Big| \int_{\mathbb{R}^n \setminus Q_m^*} K(x, x-z)\varphi_2(z) \,\mathrm{d}z \Big| \\ &\leq C_{2N} \int_{\mathbb{R}^n \setminus Q_m^*} |x-z|^{-2N} |\varphi_2(z)| \,\mathrm{d}z \ . \end{aligned} \tag{2.85}$$

Now let $\lambda \geq \sqrt{n} + 1$. Then there exists a constant $C_{\lambda,N}$ such that

$$\frac{|x-z|^{-2N}}{(\lambda + |x-z|)^{-2N}} = \frac{(\lambda + |x-z|)^{2N}}{|x-z|^{2N}} \leq C_{\lambda,N} \tag{2.86}$$

for all $x \in Q_m$ and all $z \in \mathbb{R}^n \setminus Q_m^*$. Thus,

$$|(T_a\varphi_2)(x)| \leq C_{2N} C_{\lambda,N} \int_{\mathbb{R}^n \setminus Q_m^*} (\lambda + |x-z|)^{-2N} |\varphi_2(z)| \,\mathrm{d}z \tag{2.87}$$

for $x \in Q_m$. Note that we can give a bound on $\lambda + |x-z|$ by

$$\begin{aligned} \lambda + |x-z| &= \lambda + |x - m + m - z| \geq \lambda - |x-m| + |z-m| \\ &\geq \left(\lambda - \frac{\sqrt{n}}{2}\right) + |m-z| \geq \mu + |m-z| \end{aligned} \tag{2.88}$$

where $\mu = \sqrt{n}/2 + 1$. We can now give an estimate on J_m: by Minkowski's inequality and by Hölder's inequality

$$\begin{aligned}
J_m^{1/2} &= \left(\int_{Q_m} |(T_a\varphi)(x)|^2 \, \mathrm{d}x \right)^{1/2} \\
&\le C_{2N} C_{\lambda,N} \left\{ \int_{Q_m} \left| \int_{\mathbb{R}_n \setminus Q_m^*} \frac{(\mu + |x-z|)^{-N} |\varphi_2(z)|}{(\mu + |m-z|)^N} \, \mathrm{d}z \right|^2 \mathrm{d}x \right\}^{1/2} \\
&\le C_{2N} C_{\lambda,N} \int_{\mathbb{R}^n \setminus Q_m^*} \left\{ \int_{Q_m} \frac{(\mu + |x-z|)^{-2N} |\varphi_2(z)|^2}{(\mu + |m-z|)^{2N}} \, \mathrm{d}x \right\}^{1/2} \mathrm{d}z \\
&= C_{2N} C_{\lambda,N} \int_{\mathbb{R}^n \setminus Q_m^*} \frac{|\varphi_2(z)|}{(\mu + |m-z|)^{N/2}} \frac{1}{(\mu + |m-z|)^{N/2}} \\
&\qquad \left\{ \int_{Q_m} (\mu + |x-z|)^{-2N} \, \mathrm{d}x \right\}^{1/2} \mathrm{d}z \\
&\le C_{2N} C_{\lambda,N} \left\{ \int_{\mathbb{R}^n \setminus Q_m^*} \frac{|\varphi_2(z)|^2}{(\mu + |m-z|)^N} \, \mathrm{d}z \right\}^{1/2} \\
&\qquad \left\{ \int_{\mathbb{R}^n \setminus Q_m^*} \frac{1}{(\mu + |m-z|)^N} \, \mathrm{d}z \right\}^{1/2} \left\{ \int_{Q_m} (\mu + |x-z|)^{-2N} \, \mathrm{d}x \right\}^{1/2} .
\end{aligned} \tag{2.89}$$

Hence, we can find a different constant $C_{\lambda,N}$ such that

$$J_m \le C_{\lambda,N} \int_{\mathbb{R}^n \setminus Q_m^*} \frac{|\varphi_2(z)|^2}{(\mu + |m-z|)^N} \, \mathrm{d}z \, . \tag{2.90}$$

Summing over all lattice points m we find that

$$\begin{aligned}
C_{\lambda,N} \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n \setminus Q_m^*} \frac{|\varphi_2(z)|^2}{(\mu + |m-z|)^N} \, \mathrm{d}z & \\
&\le C_{\lambda,N} \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n \setminus Q_m} \frac{|\varphi_2(z)|^2}{(\mu + |m-z|)^N} \, \mathrm{d}z \\
&= C_{\lambda,N} \sum_{m \in \mathbb{Z}^n} \sum_{l \ne m} \int_{Q_l} \frac{|\varphi_2(z)|^2}{(\mu + |m-z|)^N} \, \mathrm{d}z \, .
\end{aligned} \tag{2.91}$$

Now for all $m, l \in \mathbb{Z}^n$ it holds that

$$\begin{aligned}
\mu + |m-z| = \mu + |m-l+l-z| &\ge \mu + |m-l| - |l-z| \\
&\ge \mu + |m-l| \ge 1 + |m-l| \, .
\end{aligned} \tag{2.92}$$

Hence, by the positivity of the summands

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}^n}\sum_{l\neq m}\int_{Q_l}\frac{|\varphi_2(z)|^2}{(\mu+|m-z|)^N}\,\mathrm{d}z \\
&\quad\leq \sum_{m\in\mathbb{Z}^n}\sum_{l\neq m}\frac{1}{(1+|m-l|)^N}\int_{Q_l}|\varphi_2(z)|^2\,\mathrm{d}z \\
&\quad\leq \sum_{m\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\frac{1}{(1+|m-l|)^N}\int_{Q_l}|\varphi_2(z)|^2\,\mathrm{d}z \\
&\quad= \sum_{l\in\mathbb{Z}^n}\int_{Q_l}|\varphi_2(z)|^2\,\mathrm{d}z\sum_{m\in\mathbb{Z}^n}\frac{1}{(1+|m-l|)^N} \\
&\quad= \sum_{l\in\mathbb{Z}^n}\int_{Q_l}|\varphi_2(z)|^2\,\mathrm{d}z\sum_{m'\in\mathbb{Z}^n}\frac{1}{(1+|m'|)^N} \\
&\quad= \sum_{m'\in\mathbb{Z}^n}\frac{1}{(1+|m'|)^N}\int_{\mathbb{R}^n}|\varphi_2(z)|^2\,\mathrm{d}z\ .
\end{aligned}
\tag{2.93}
$$

Hence, we can conclude that

$$
\begin{aligned}
&\|(T_a\varphi)\|_2^2 = \int_{\mathbb{R}^n}|(T_a\varphi)(x)|^2\,\mathrm{d}x \leq \sum_{m\in\mathbb{Z}^n} I_m \\
&\leq 2D_N^2\sum_{m\in\mathbb{Z}^n}\int_{Q_m^{**}}|\varphi(x)|^2\,\mathrm{d}x + 2C_{\lambda,N}\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n\setminus Q_m^*}\frac{|\varphi_2(z)|^2}{(\mu+|m-z|)^N}\,\mathrm{d}z \\
&\leq D\int_{\mathbb{R}^n}|\varphi(x)|^2\,\mathrm{d}x + 2C_{\lambda,N}\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n\setminus Q_m^*}\frac{|\varphi_2(z)|^2}{(\mu+|m-z|)^N}\,\mathrm{d}z \\
&\leq \left\{D + 2C_{\lambda,N}\sum_{m'\in\mathbb{Z}^n}\frac{1}{(1+|m'|)^N}\right\}\int_{\mathbb{R}^n}|\varphi(x)|^2\,\mathrm{d}x = C\|\varphi\|_2^2\ ,
\end{aligned}
\tag{2.94}
$$

which means that T_a is a bounded linear operator, if we can establish the bound $|\hat{a}(\lambda,\xi)| \leq C_N(1+|\lambda|)^{-N}$. ∎

The following lemma finishes the proof for Theorem 2.3.

Lemma 2.17. For any integer N, and any multi-index α it holds that

$$
|D_\xi^\alpha \hat{a}_m(\lambda,\xi)| \leq C_N(1+|\lambda|)^{-N}(1+|\xi|)^{-|\alpha|}\ . \tag{2.95}
$$

Proof. By Leibniz formula and integration by parts

$$
\begin{aligned}
&(-i\lambda)^\beta(D_\xi^\alpha\hat{a}_m(\lambda,\xi)) \\
&= (-1)^{|\beta|}(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-ix\cdot\lambda}\partial_x^\beta\left(\eta(x-m)D_\xi^\alpha a(x,\xi)\right)\,\mathrm{d}x \\
&= (-1)^{|\beta|}(2\pi)^{-n/2} \\
&\quad\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}\int_{\mathbb{R}^n}e^{-ix\cdot\lambda}\partial_x^{\beta'}(\eta)(x-m)\left(\partial_x^{\beta-\beta'}D_\xi^\alpha a(x,\xi)\right)\,\mathrm{d}x\ .
\end{aligned}
\tag{2.96}
$$

Hence, because a is a symbol of order 0 it follows that

$$|(-i\lambda)^{\beta} D_{\xi}^{\alpha} \hat{a}_m(\lambda, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|} \ , \tag{2.97}$$

from which the statement of the lemma follows. ■

The proof of Theorem 2.3 holds for general $p \in (1, \infty)$ after establishing the fact that T_λ is a bounded linear operator, but requires Hörmander's inequality, see Theorem 2.5 in [Hör60].

2.3 Elliptic pseudodifferential operators and the parametrix

In this section we look at elliptic pseudodifferential operators. For this class of pseudodifferential operators, we will show that there exists an approximate inverse. This approximate inverse will show that solutions to elliptic pseudodifferential equations $T_a u = f$ are smooth if f is smooth.

Definition 2.18 (Elliptic pseudodifferential operator)**.** A symbol $a(x, \xi) \in \mathcal{S}^k$ of order k is called elliptic if there are positive constants C, R, which are independent of x, such that

$$|a(x, \xi)| \geq C(1 + |\xi|)^k \quad \text{for } \xi \geq R. \tag{2.98}$$

A pseudodifferential operator is called elliptic if its symbol is elliptic.

The following lemma shows that only the principal symbol needs to be elliptic for the whole symbol to be elliptic.

Lemma 2.19. If $b(x, \xi) \in \mathcal{S}^k$ has an asymptotic expansion $b \sim \sum_{l=0}^{\infty} b_l$, then b is elliptic if and only if the principal symbol b_0 is elliptic.

Proof. Since b_0 is elliptic, there are constants $C_0, R_0 > 0$, such that we have $|b_0(x, \xi)| > C_0(1 + |\xi|)^k$ for $|\xi| > R_0$. Hence, because b is a pseudodifferential operator, for $|\xi| > R_0$ we have

$$\begin{aligned} |b(x, \xi)| = \left| \sum_{l=0}^{\infty} b_l(x, \xi) \right| &= \left| b_0(x, \xi) + \sum_{l=1}^{\infty} b_l(x, \xi) \right| \\ &\geq |b_0(x, \xi)| - \left| \sum_{l=1}^{\infty} b_l(x, \xi) \right| \\ &\geq C_0(1 + |\xi|)^k - C_0'(1 + |\xi|)^{k_1} \\ &\quad = (1 + |\xi|)^k (C_0 - C_0'(1 + |\xi|)^{k_1 - k}) \ . \end{aligned} \tag{2.99}$$

Because $k_1 - k < 0$, there is a positive constant R_1, such that

$$C_0 - C_0'(1 + |\xi|)^{k_1 - k} \geq \frac{C_0}{2} \quad \text{for } |\xi| > R_1. \tag{2.100}$$

Now set $R = \max\{R_0, R_1\}$. For $|\xi| > R$ it follows that

$$|b(x,\xi)| \geq C(1 + |\xi|)^k \tag{2.101}$$

and hence $b(x,\xi)$ is elliptic.

Now assume that $b(x,\xi)$ is elliptic, then there are constants C_0, R_0 such that for $|\xi| > R_0$

$$\begin{aligned} C_0(1+|\xi|)^k \leq |b(x,\xi)| = \left|\sum_{l=0}^{\infty} b_l(x,\xi)\right| &\leq |b_0(x,\xi)| + \left|\sum_{l=1}^{\infty} b_l(x,\xi)\right| \\ &\leq |b_0(x,\xi)| + C'(1+|\xi|)^{k_1} \ . \end{aligned} \tag{2.102}$$

Now there is some R_1 such that for $|\xi| > R_1$ it holds that $C'(1+|\xi|)^{k_1}$ can be bounded by $\frac{1}{2}C_0(1+|\xi|)^k$. Now for $R = \max\{R_0, R_1\} > 0$, $C = \frac{1}{2}C_0 > 0$ and all $|\xi| > R$ it holds that $|b_0(x,\xi)| \geq C(1+|\xi|)^k$. Thus, the principal symbol is elliptic, and the lemma is proved. ■

The motivation for the following theorem arises from the elliptic pseudodifferential equation

$$T_a u = f \ , \tag{2.103}$$

with a an elliptic symbol of order k and $f \in L^2$ a function. We want to find the function u such that $T_a u = f$. The following theorem will show that there is a u such that

$$T_a u - f \in C^\infty(\mathbb{R}^n) \ . \tag{2.104}$$

In a more general sense we want to invert the operator T_a by some operator T_b. We call the symbol b attached to this operator the parametrix of a.

Theorem 2.6. *If a is an elliptic symbol of order k and T_a is the corresponding pseudodifferential operator, then there is an elliptic symbol b of order $-k$ with an asymptotic expansion given by*

$$b \sim \sum_{l=0}^{\infty} b_{-k-l} \tag{2.105}$$

where b_{-k-l} is a symbol of order $-k-l$, such that there are smoothing operators K_1 and K_2 such that

$$\begin{aligned} T_a \circ T_b &= I + K_1 \ , \quad \textit{and} \\ T_b \circ T_a &= I + K_2 \ . \end{aligned} \tag{2.106}$$

Proof. Giving an asymptotic expansion of b is sufficient, since Proposition 2.8 shows the existence of a symbol, given elements of an asymptotic expansion, and Proposition 2.5 shows that if b' is a different elliptic pseudodifferential operator with an identical asymptotic expansion, then $T_{b-b'}$ is infinitely smoothing and may be absorbed in the operators K_1 and K_2.

The product of the symbols b and a is given by Equation (2.21):

$$(b\sharp a)(x,\xi) \sim \sum_{\alpha}\sum_{j=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi)) \ . \tag{2.107}$$

Observe that the finite sums

$$\begin{aligned}&\sum_{|\alpha|<N}\sum_{j=0}^{N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi)) \\ &= a(x,\xi)b_{-k}(x,\xi) + \sum_{j=1}^{N-1}\sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi))\end{aligned} \tag{2.108}$$

are equal to

$$\begin{aligned}&a(x,\xi)b_{-k}(x,\xi)+ \\ &\quad\sum_{l=1}^{N-1}\left(a(x,\xi)b_{-k-l} + \sum_{\substack{|\alpha|+j=l\\ 0\le j<l}} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi))\right) \\ &\qquad + \sum_{\substack{|\alpha|+j\ge N\\ |\alpha|<N\\ j<N}} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi))\end{aligned} \tag{2.109}$$

with

$$\sum_{\substack{|\alpha|+j\ge N\\ |\alpha|<N\\ j<N}} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi)) \in \mathcal{S}^{-N} \ . \tag{2.110}$$

Then set b_{-k} to be the symbol of order $-k$ given by

$$b_{-k}(x,\xi) = \frac{1-\phi(\xi)}{a(x,\xi)} = \frac{\psi(\xi)}{a(x,\xi)} \tag{2.111}$$

where $0 \le \phi(\xi) \le 1 \in C_0^\infty(\mathbb{R}^n)$ is a compactly supported smooth function such that

$$\phi(\xi) = \begin{cases} 1 & |\xi| < R \\ 0 & |\xi| \ge 2R. \end{cases} \tag{2.112}$$

Then b_{-k} is a symbol of order $-k$. Indeed, since $a(x,\xi)$ is elliptic, for β a multi-index

$$\begin{aligned}|D_\xi^\beta b_{-k}(x,\xi)| &= \begin{cases} |D_\xi^\beta \frac{1}{a(x,\xi)}| & |\xi| \ge 2R \\ |D_\xi^\beta \frac{\psi(\xi)}{a(x,\xi)}| & R < |\xi| \le 2R \\ 0 & |\xi| < R \end{cases} \\ &\le \begin{cases} \frac{1}{C}(1+|\xi|)^{-k-|\beta|} & |\xi| \ge 2R \\ \left|\frac{1}{C}\sum_{\gamma\le\beta}\binom{\beta}{\gamma} D_\xi^{\beta-\gamma}\psi(\xi)(1+|\xi|)^{-k-|\gamma|}\right| & R < |\xi| \le 2R \\ 0 & |\xi| < R \ . \end{cases}\end{aligned} \tag{2.113}$$

Because the set $|\xi| \leq 2R$ is compact, it is possible to find a constant C' such that

$$\left| \frac{1}{C} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D_\xi^{\beta-\gamma} \psi(\xi)(1+|\xi|)^{-k-|\gamma|} \right| \leq C'(1+|\xi|)^{-k-|\beta|} \tag{2.114}$$

for all $|\xi| \leq 2R$. Then for $D = \max\{1/C, C'\}$ it follows that

$$|D_\xi^\beta b_{-k}(x,\xi)| \leq D(1+|\xi|)^{-k-|\beta|} \ . \tag{2.115}$$

Now inductively set b_{-k-l} to be the symbol of order $-k-l$ given by

$$b_{-k-l}(x,\xi) = -\left(\sum_{\substack{|\alpha|+j=l \\ 0 \leq j < l}} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi)) \right) b_{-k} \ . \tag{2.116}$$

Then b_{-k-l} is indeed a symbol, since it is the derivative, sum and product of symbols, each of which (inductively) satisfy the symbol properties from Definition 2.1. Then by Equations (2.107) and (2.116)

$$(b_{-k} \sharp a)(x,\xi) = 1 - \phi(\xi) \in \mathcal{S}^0 \tag{2.117}$$

and

$$\begin{aligned} &\sum_{j=0}^{N-1} (b_{-k-j} \sharp a)(x,\xi) - 1 \\ &= \sum_{\substack{|\alpha|+j \geq N \\ j<N}} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_x^\alpha a(x,\xi))(\partial_\xi^\alpha b_{-k-j}(x,\xi)) \in \mathcal{S}^{-N} \ , \end{aligned} \tag{2.118}$$

hence

$$(b \sharp a)(x,\xi) = \left(\sum_{l=0}^\infty b_{-k-l} \sharp a \right) = 1 + R(x,\xi) \quad \text{with} \quad R(x,\xi) \in \mathcal{S}^{-\infty} \ , \tag{2.119}$$

such that $T_b \circ T_a = T_{b\sharp a} = I + K_2$, where K_2 is infinitely smoothing.

It is still required to show that b is elliptic. By Lemma 2.19 it is sufficient to show that b_{-k} is elliptic. This is indeed the case, since for $|\xi| \geq 2R$

$$|b_{-k}(x,\xi)| = \left| \frac{1}{a(x,\xi)} \right| \geq \frac{1}{C_{0,0}} (1+|\xi|)^{-k} \tag{2.120}$$

because $a(x,\xi)$ is a symbol. ■

The construction in the proof creates a parametrix $b(x,\xi)$ in which each of the symbols b_{-k-l} in the asymptotic expansion is not homogeneous, if $a(x,\xi)$ is not homogeneous. The next proposition aims to solve this.

Proposition 2.20. Suppose that $a(x,\xi) = a_k(x,\xi) + a_{k-1}(x,\xi)$ is an elliptic symbol of order k, and both $a_k(x,\xi)$ and $a_{k-1}(x,\xi)$ are homogeneous of degrees k and $k-1$ respectively, then there is a parametrix $c \sim \sum_{l=0}^{\infty} c_{-k-l}$, such that each b'_{-k-l} is homogeneous of degree $-k-l$.

Proof. Start by creating the parametrix b_{-k-l} of the principal symbol a_k. Since a_k is homogeneous of degree k, each of the symbols b_{-k-l} are homogeneous of degree $-k-l$. Now define the symbols

$$c_{-k-l} = b_{-k-l} + b'_{-k-l} \tag{2.121}$$

where b'_{-k-l} is defined by

$$\begin{aligned} b'_{-k-l} = &-\sum_{m=0}^{l-1}((a_{k-1}\sharp b_{-k-m})_{-l} + (a_k\sharp b'_{-k-m})_{-l})b_{-k} \\ &-\sum_{m=0}^{l-2}(a_{k-1}\sharp b'_{-k-m})_{-l}b_{-k}\ . \end{aligned} \tag{2.122}$$

Then c_{-k-l} is by construction homogeneous of degree $-k-l$, and is a parametrix of $a(x,\xi) = a_k(x,\xi) + a_{k-1}(x,\xi)$ also by construction. ■

We will use this construction in Chapter 3 to compute the leading order terms ($l = 0, l = 1, l = 2$) of the parametrix of the Laplace-Beltrami operator.

2.4 Sobolev spaces and Elliptic regularity

In this section, we give a more general definition of the Sobolev spaces. We show that pseudodifferential operators are bounded linear operators between Sobolev spaces. This gives an immediate application of pseudodifferential operators.

Definition 2.21 (Sobolev spaces)**.** For s a real number, define the Sobolev space $H^s(\mathbb{R}^n)$ of order s as the space of tempered distributions f such that

$$(1 + |\xi|^2)^{s/2}\hat{f}(\xi) \in L^2(\mathbb{R}^n)\ . \tag{2.123}$$

Equivalently H^s is the norm completion of the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_s = \left(\int_{\mathbb{R}^n}(1 + |\xi|^2)^s|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\ . \tag{2.124}$$

Furthermore, there is an inner product $(\cdot,\cdot)$ on H^s such that the norm induced from the inner product coincides with the norm given in Equation (2.124).

Notice the following things about Sobolev spaces.

Remark 2.22. The function $(1 + |\xi|^2)^{s/2}$ is a symbol of order s. Set J_{-s} as the pseudodifferential operator belonging to this symbol. Then H^s is the set of tempered distributions f such that $J_{-s}f \in L^2$.

Remark 2.23. The space H^0 is L^2.

Remark 2.24. If s is an integer k, then the Sobolev space $H^s(\mathbb{R}^n)$ is identical to the classical Sobolev space of equivalence classes of k times differentiable functions in L^2 with the norm given by

$$\|f\|_{H^k} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_2 \ . \tag{2.125}$$

Using Theorem 2.3 we can give the following proposition.

Proposition 2.25. Let $a \in S^k$ a symbol, then $T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to a bounded linear operator $T_a : H^s \to H^{s-k}$.

Proof. First notice that $J_{s-k}T_aJ_s \in \mathcal{S}^0$, and hence by Theorem 2.3, it follows that $J_{s-k}T_aJ_s : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to a bounded linear operator $L^2 \to L^2$.

Then by definition of the Sobolev spaces it follows that $T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to a bounded linear operator $T_a : H^s \to H^{s-k}$. ■

Using the previous sections we can give a formulation of the elliptic regularity.

Theorem 2.7 (Elliptic regularity)**.** *Suppose that $a(x, \xi) \in \mathcal{S}^k$ is an elliptic symbol. Then for every $f \in H^s$ there is an element $u \in H^{s+k}$ such that*

$$T_a u - f \in C^\infty(\mathbb{R}^n) \ . \tag{2.126}$$

Furthermore, the singular supports of u and f are equal.

Proof. Using the parametrix $b(x, \xi) \in \mathcal{S}^{-k}$ of $a(x, \xi) \in \mathcal{S}^k$ we are able to find an element $u = T_b f \in H^{s+k}$ such that

$$T_a u - f = (T_a \circ T_b)f - f = K_1 f \tag{2.127}$$

is smooth by Theorem 2.6.

To show that the singular supports are equal, we use Lemma 2.15. Indeed, we have the inclusions

$$\operatorname{sing\,supp} f = \operatorname{sing\,supp} T_a u \subseteq \operatorname{sing\,supp} u \tag{2.128}$$

and

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} T_b f \subseteq \operatorname{sing\,supp} f \ , \tag{2.129}$$

which proves the theorem. ■

3 Inverting the Laplace-Beltrami Operator

In this chapter we will look at pseudodifferential operators on Riemannian manifolds. In particular, we look at the extension of the Laplacian on Euclidean space to a pseudodifferential operator called the Laplace-Beltrami operator on a Riemannian manifold. We show that the Laplace-Beltrami operator is elliptic, and calculate its parametrix. Next we introduce normal local coordinates on a Riemannian manifold, and give some properties, which we use to reduce the parametrix of the Laplace-Beltrami operator.

3.1 The Laplace-Beltrami operator

On a Riemannian smooth manifold (M, g), there is a pseudodifferential operator, the Laplace-Beltrami operator.

Definition 3.1 (Laplace-Beltrami operator)**.** Let (M, g) be a Riemannian manifold, i.e. a smooth manifold M with a positive definite symmetric bilinear form $g_x : T_xM \times T_xM \to \mathbb{R}$, depending smoothly on x. Let $(U_\alpha, \varphi_\alpha)$ be a maximal atlas. In local coordinates the Laplace-Beltrami operator is defined as

$$\Delta_g(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\sqrt{|g(x)|}} \partial_{x_i} \left(\sqrt{|g(x)|} g^{ij}(x) \partial_{x_j} \right) , \tag{3.1}$$

where g^{ij} denotes the inverse matrix of the local Riemannian metric form g_{ij}, and $|g(x)| = |\det g_{ij}(x)|$. For convenience the dependence on x of the metric form g_{ij} will be omitted, whenever it is clear.

In this thesis we will assume that $M = \mathbb{R}^n$ and furthermore we will assume that there is some constant D, such that $g_{ij}(x) = \delta_{ij}$ whenever $|x| > D$.

Proposition 3.2 (Symbol of the Laplace-Beltrami operator)**.** The symbol of the Laplace-Beltrami operator is given by

$$\sigma(\Delta_g)(x, \xi) = -i \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial_{x_i} |g|}{2|g|} g^{ij} + \partial_{x_i} g^{ij} - i g^{ij} \xi_i \right) \xi_j , \tag{3.2}$$

equivalently

$$\sigma(\Delta_g)(x,\xi) = -i\left(\frac{1}{2|g|}\left\langle \nabla|g|, g^{ij}\xi\right\rangle + \left\langle \nabla\cdot g^{ij},\xi\right\rangle - i\left\langle \xi, g^{ij}\xi\right\rangle\right) , \tag{3.3}$$

where $\nabla|g|$ denotes the gradient of $|g|$ and $\nabla\cdot g^{ij}$ denotes the row-wise divergence of g^{ij}.

Proof. From Equation (3.1) it follows by Leibniz rule for differentiation that

$$\begin{aligned}\Delta_g(x) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{|g(x)|}}\partial_{x_i}\left(\sqrt{|g(x)|}g^{ij}(x)\partial_{x_j}\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{|g(x)|}}\Big(\partial_{x_i}\left(\sqrt{|g|}\right)g^{ij}\\ &\qquad\qquad + \sqrt{|g|}\partial_{x_i}g^{ij} + \sqrt{|g|}g^{ij}\partial_{x_i}\Big)\partial_{x_j}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial_{x_i}|g|}{2|g|}g^{ij} + \partial_{x_i}g^{ij} + g^{ij}\partial_{x_i}\right)\partial_{x_j} .\end{aligned} \tag{3.4}$$

Replacing the derivatives ∂_{x_i} and ∂_{x_j} by their symbols $-i\xi_i$ and $-i\xi_j$, respectively, Equation (3.2) follows. ■

Proposition 3.3. The Laplace-Beltrami operator is a pseudodifferential operator of order 2. Furthermore, it is elliptic.

Proof. We have to show that there exists constants $C_{\alpha,\beta}, C', R$ such that

(a) For all $x,\xi\in\mathbb{R}^n$ and all multi-indices α,β it holds that

$$|D_x^\alpha D_\xi^\beta \sigma(\Delta_g)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{2-|\beta|} , \tag{3.5}$$

where the constant $C_{\alpha,\beta}$ depends only on the multi-indices α,β.

(b) For all $x\in\mathbb{R}^n$, all $|\xi|\ge R$

$$|\sigma(\Delta_g)(x,\xi)| \ge C'(1+|\xi|)^2 . \tag{3.6}$$

For part (a), notice that since $g_{ij}(x)=\delta_{ij}$, whenever $|x|>D$, we have that

$$\begin{aligned}&|D_x^\alpha D_\xi^\beta \sigma(\Delta_g)(x,\xi)|\\ &\quad\le \left|D_x^\alpha\left(\frac{\partial_{x_i}|g|}{2|g|}g^{ij} + \partial_{x_i}g^{ij}\right)D_\xi^\beta\xi_j\right| + |D_x^\alpha g^{ij} D_\xi^\beta(\xi_i\xi_j)|\\ &\quad\le C_{\alpha,\beta}(1+|\xi|)^{2-|\beta|} ,\end{aligned} \tag{3.7}$$

because derivatives in x outside the compact set $\{|x| \leq D\}$ are 0, and since inside the compact set there is a maximum attained.

For part (b), notice that by Equation (3.3) automatically

$$\begin{aligned} |\sigma(\Delta_g)(x,\xi)| &= \left| i\frac{1}{2|g|} \left\langle \nabla |g|, g^{ij}\xi \right\rangle + i \left\langle \nabla \cdot g^{ij}, \xi \right\rangle + \left\langle \xi, g^{ij}\xi \right\rangle \right| \\ &\geq | \left\langle \xi, g^{ij}\xi \right\rangle | > \tilde{C}(x)|\xi|^2 \qquad \tilde{C}(x) > 0 \ , \end{aligned} \tag{3.8}$$

since g^{ij} defines for each $x \in \mathbb{R}^n$ an equivalent norm. Set $\hat{C} = \inf_{x\in\mathbb{R}^n} \tilde{C}(x) > 0$, and set $R > 0$, such that $|\sigma(\Delta_g)(x,\xi)| \geq \hat{C}|\xi|^2 > 2$ for $|\xi| \geq R$. Then there is indeed a constant $C' > 0$, such that for all $|\xi| > R$ it follows that for every $x \in \mathbb{R}^n$

$$|\sigma(\Delta_g)(x,\xi)| \geq C'(1 + |\xi|)^2 \ . \tag{3.9}$$

Thus, $\sigma(\Delta_g)$ is an elliptic symbol. Of course by Lemma 2.19 it would have been sufficient to check if the principal symbol was elliptic. ∎

3.2 Parametrix of the Laplace-Beltrami operator

In this section we give a method for computing the parametrix of the Laplace-Beltrami operator, such that each term of the parametrix is homogeneous in the frequency variable.

Theorem 3.1. *It is possible to write the symbols in the asymptotic expansion of the parametrix of the Laplace-Beltrami operator Δ_g from Definition 3.1 in terms of homogeneous symbols.*

Proof. Using Proposition 2.20 we will construct the parametrix in terms of homogeneous symbols. The principal symbol of Δ_g is $\sigma_2 = -\langle \xi, g^{ij}\xi\rangle$. Furthermore, define σ_1 to be the difference $\sigma(\Delta_g) - \sigma_2$.

The symbols of the parametrix of the principal symbol $\sigma_2 = -\langle \xi, g^{ij}\xi\rangle$ of $\sigma(\Delta_g)$ can be found using Equation (2.116), and has terms

$$\begin{aligned} \tau_{-2} &= \frac{1-\phi(\xi)}{-\langle \xi, g^{ij}\xi\rangle} \\ \tau_{-3} &= -i \sum_{k_1=1}^{n} \left\langle \xi, \partial_{x_{k_1}} g^{ij}\xi \right\rangle \left(\frac{\langle \xi, g^{ij}\xi\rangle \partial_{\xi_{k_1}}\psi(\xi) - 2\psi(\xi)\langle e_{k_1}, g^{ij}\xi\rangle}{\langle \xi, g^{ij}\xi\rangle^2} \right) \tau_{-2} \ . \end{aligned} \tag{3.10}$$

However, if one only looks at $|\xi| > R$

$$\begin{aligned} \tau_{-2} &= \frac{1}{-\langle \xi, g^{ij}\xi\rangle} \\ \tau_{-3} &= -i \sum_{k_1=1}^{n} \left\langle \xi, \partial_{x_{k_1}} g^{ij}\xi \right\rangle \left(\frac{2\langle e_{k_1}, g^{ij}\xi\rangle}{\langle \xi, g^{ij}\xi\rangle^3} \right) \ . \end{aligned} \tag{3.11}$$

Now we give a correction term to τ_{-3} such that

$$\tau_{-3} \cdot \sigma_2 = -\tau_{-2} \cdot \sigma_1 \tag{3.12}$$

with the correction term to τ_{-3} given by

$$-\frac{i}{\langle \xi, g^{ij}\xi\rangle^2}\left(\frac{1}{2|g|}\left\langle \nabla|g|, g^{ij}\xi\right\rangle + \left\langle \nabla \cdot g^{ij}, \xi\right\rangle\right) \ . \tag{3.13}$$

For the next term, τ_{-4}, we find that

$$\begin{aligned}
\tau_{-4} = \Bigg[&\sum_{k_2=1}^{n}\sum_{k_3=k_2+1}^{n} -\left\langle \xi, \partial_{x_{k_2}}\partial_{x_{k_3}} g^{ij}\xi\right\rangle \\
&\left(\frac{(2\langle e_{k_2}, g^{ij}e_{k_3}\rangle)}{\langle \xi, g^{ij}\xi\rangle^3} - \frac{8\langle e_{k_3}, g^{ij}\xi\rangle\langle \xi, g^{ij}\xi\rangle(\langle e_{k_2}, g^{ij}\xi\rangle)}{\langle \xi, g^{ij}\xi\rangle^5}\right) \\
&+\sum_{k_2=1}^{n}\frac{-1}{2}\left\langle \xi, \partial_{x_{k_2}}\partial_{x_{k_2}} g^{ij}\xi\right\rangle \\
&\left(\frac{(2\langle e_{k_2}, g^{ij}e_{k_2}\rangle)}{\langle \xi, g^{ij}\xi\rangle^3} - \frac{8\langle e_{k_2}, g^{ij}\xi\rangle\langle \xi, g^{ij}\xi\rangle(\langle e_{k_2}, g^{ij}\xi\rangle)}{\langle \xi, g^{ij}\xi\rangle^5}\right) \\
&-\sum_{k_4=1}^{n}\left\langle \xi, \partial_{x_{k_4}} g^{ij}\xi\right\rangle\Bigg\{\sum_{k_1=1}^{n} 4\left\langle e_{k_4}, \partial_{x_{k_1}} g^{ij}\xi\right\rangle\frac{\langle e_{k_1}, g^{ij}\xi\rangle}{\langle \xi, g^{ij}\xi\rangle^4} + \\
&\left\langle \xi, \partial_{x_{k_1}} g^{ij}\xi\right\rangle\left(\frac{2\langle e_{k_1}, g^{ij}e_{k_4}\rangle}{\langle \xi, g^{ij}\xi\rangle^4} - 12\frac{\langle e_{k_1}, g^{ij}\xi\rangle\langle e_{k_4}, g^{ij}\xi\rangle}{\langle \xi, g^{ij}\xi\rangle^5}\right) \\
&+\frac{1}{\langle \xi, g^{ij}\xi\rangle^3}\left(\frac{1}{2|g|}\left\langle \nabla|g|, g^{ij}e_{k_4}\right\rangle + \left\langle \nabla \cdot g^{ij}, e_{k_4}\right\rangle\right. \\
&\left.-\frac{4\langle e_{k_4}, g^{ij}\xi\rangle}{\langle \xi, g^{ij}\xi\rangle}\left(\frac{1}{2|g|}\left\langle \nabla|g|, g^{ij}\xi\right\rangle + \left\langle \nabla \cdot g^{ij}, \xi\right\rangle\right)\right)\Bigg\}\Bigg] \ ,
\end{aligned} \tag{3.14}$$

for $|\xi| > R$. Next we correct for the terms of order -2 in the composition calculus of τ_{-2} and τ_{-3} with σ_1 by adding

$$\frac{1}{\langle \xi, g^{ij}\xi\rangle}\left(\sigma_1 \cdot \tau_{-3} - i\nabla_x\sigma_1 \cdot \nabla_\xi\tau_{-2}\right) \ . \tag{3.15}$$

Inductively one is able to construct the parametrix of $\sigma(\Delta_g)$ where each symbol τ_{-2-k} is homogeneous of degree $-2-k$. ■

3.3 Normal coordinates on a Riemannian manifold

In this section we define normal local coordinates on Riemannian manifolds. We will see that locally the metric tensor $g_{ij}(x) = \delta_{ij} + \mathcal{O}(|x|^2)$. These results we will use in the next section to reduce the terms in the parametrix. As a result we will see that $\tau_{-3}(0,\xi) = 0$. This section uses material which can be found in more detail in [Lee18]. From this section forward we will use the Einstein summation convention, and assume the reader has familiarity with it.

Christoffel symbols and Curvature

Definition 3.4 (Christoffel symbols)**.** We define

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \tag{3.16}$$

to be the Christoffel symbols of a metric tensor g_{ij}.

Using the Christoffel symbol, we define the Riemann curvature tensor.

Definition 3.5 (Riemann, Ricci and Scalar Curvature tensors)**.** The Riemann curvature R_{ijkl} is defined to be

$$R_{ijkl} = g_{lm} \left(\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} + \Gamma^p_{ik}\Gamma^m_{jp} - \Gamma^p_{jk}\Gamma^m_{ip}\right) \ . \tag{3.17}$$

Furthermore, define the Ricci curvature tensor to be

$$R_{jl} = g^{ik} R_{ijkl} \ . \tag{3.18}$$

Finally, set R to be the scalar curvature, defined by

$$R = g^{jl} R_{jl} \ . \tag{3.19}$$

Different authors use different conventions for the Riemann curvature. Here we have chosen the convention, such that the unit 2-sphere has a constant positive scalar curvature of 2.

The Riemann curvature has symmetries, which we will use to reduce this tensor.

Proposition 3.6 (Symmetries of the Riemann curvature tensor)**.** The following identities hold for the Riemann curvature tensor

(a) Skew-symmetry in the first two variables: $R_{ijkl} = -R_{jikl}$

(b) Skew-symmetry in the last two variables: $R_{ijkl} = -R_{ijlk}$

(c) Interchange symmetry: $R_{ijkl} = R_{klij}$

(d) Bianchi identity: $R_{ijkl} + R_{jkil} + R_{kijl} = 0$.

Proof. (a) Follows from writing out the definition of the Riemann curvature tensor.

(b) The proof for this symmetry uses the *metric compatibility*

$$\partial_k g_{ij} = \Gamma^l_{kj} g_{il} + \Gamma^l_{ki} g_{jl} \tag{3.20}$$

of the Christoffel symbol. Indeed, by writing out the definition of Γ^l_{ki} and Γ^l_{kj} we see that

$$\begin{aligned} \Gamma^l_{ki} g_{lj} &= \frac{1}{2} g^{lm} \left(\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ik} \right) g_{lj} \\ &= \frac{1}{2} \delta^m_j \left(\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ik} \right) \\ &= \frac{1}{2} \left(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik} \right) \end{aligned} \tag{3.21}$$

and that

$$\begin{aligned} \Gamma^l_{kj} g_{li} &= \frac{1}{2} g^{lm} \left(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk} \right) g_{li} \\ &= \frac{1}{2} \delta^m_i \left(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk} \right) \\ &= \frac{1}{2} \left(\partial_k g_{ij} - \partial_i g_{kj} + \partial_j g_{ik} \right) \ . \end{aligned} \tag{3.22}$$

Hence, it follows that

$$\begin{aligned} R_{ijkl} + R_{ijlk} = g_{lm} &\left(\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} + \Gamma^p_{ik} \Gamma^m_{jp} - \Gamma^p_{jk} \Gamma^m_{ip} \right) \\ &+ g_{km} \left(\partial_j \Gamma^m_{il} - \partial_i \Gamma^m_{jl} + \Gamma^p_{il} \Gamma^m_{jp} - \Gamma^p_{jl} \Gamma^m_{ip} \right) \\ = g_{lm} &\left(\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} \right) + g_{km} \left(\partial_j \Gamma^m_{jl} - \partial_i \Gamma^m_{jl} \right) \\ + 2\Gamma^m_{il}&(\partial_j g_{km}) - 2\Gamma^m_{jk}(\partial_i g_{lm}) + 2\Gamma^m_{ik}(\partial_j g_{lm}) - 2\Gamma^m_{jl}(\partial_i g_{km}) \\ + \Gamma^p_{im}&\Gamma^m_{jl} g_{kp} - \Gamma^p_{jm}\Gamma^m_{il} g_{kp} + \Gamma^p_{im}\Gamma^m_{jk} g_{lp} - \Gamma^m_{ik}\Gamma^p_{jm} g_{lp} \ . \end{aligned} \tag{3.23}$$

By applying Leibniz rule to $\partial_j(\Gamma^m_{ik} g_{lm})$ and to $\partial_i(\Gamma^m_{jk} g_{lm})$ we notice that

$$\begin{aligned} &R_{ijkl} + R_{ijlk} \\ &\quad = \Gamma^m_{il}(\partial_j g_{km}) - \Gamma^m_{jk}(\partial_i g_{lm}) + \Gamma^m_{ik}(\partial_j g_{lm}) - \Gamma^m_{jl}(\partial_i g_{km}) \\ &\quad\quad + \Gamma^p_{im}\Gamma^m_{jl} g_{kp} - \Gamma^p_{jm}\Gamma^m_{il} g_{kp} + \Gamma^p_{im}\Gamma^m_{jk} g_{lp} - \Gamma^m_{ik}\Gamma^p_{jm} g_{lp} \\ &\quad = \Gamma^m_{il} \left(\partial_j g_{km} - \Gamma^p_{jm} g_{kp} \right) - \Gamma^m_{jk}(\partial_i g_{lm} - \Gamma^p_{im} g_{lp}) \\ &\quad\quad + \Gamma^m_{ik}(\partial_j g_{lm} - \Gamma^p_{jm} g_{lp}) - \Gamma^m_{jl} \left(\partial_i g_{km} - \Gamma^p_{im} g_{kp} \right) \\ &\quad = \Gamma^m_{il}\Gamma^p_{jk} g_{mp} - \Gamma^m_{jk}\Gamma^p_{il} g_{mp} + \Gamma^m_{ik}\Gamma^p_{jl} g_{mp} - \Gamma^m_{jl}\Gamma^p_{ik} g_{mp} = 0 \ , \end{aligned} \tag{3.24}$$

by the metric compatibility and by exchanging the indices m and p whenever it is necessary.

(c) This follows by combining the other parts of this proposition. Indeed, we see that

$$
\begin{aligned}
R_{ijkl} - R_{klij} &= R_{ijkl} + R_{likj} + R_{iklj} \\
&= R_{ijkl} - R_{lijk} - R_{ikjl} \\
&= R_{ijkl} + R_{ijlk} + R_{jlik} + R_{kjil} + R_{jikl} \\
&= R_{jlik} - R_{ijkl} + R_{kjil} \\
&= R_{jlik} - R_{ijkl} - R_{jkil} \\
&= R_{jlik} + R_{kijl} = R_{jlik} - R_{ikjl} \ .
\end{aligned} \tag{3.25}
$$

By exchanging $i \leftrightarrow j$ and $k \leftrightarrow l$ we find that

$$
R_{jilk} - R_{lkji} = -R_{jlik} + R_{ikjl} \ , \tag{3.26}
$$

such that

$$
\begin{aligned}
R_{ijkl} - R_{klij} &= R_{jilk} - R_{lkji} \\
&= -R_{jlik} + R_{ikjl} \\
&= -R_{ijkl} + R_{klij} = 0 \ .
\end{aligned} \tag{3.27}
$$

(d) Writing out the definition, we get

$$
\begin{aligned}
R_{ijkl} + R_{jkil} + R_{kijl} &= g_{lm} \left(\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} + \Gamma^p_{ik} \Gamma^m_{jp} - \Gamma^p_{jk} \Gamma^m_{ip} \right) \\
&\quad + g_{lm} \left(\partial_k \Gamma^m_{ji} - \partial_j \Gamma^m_{ki} + \Gamma^p_{ji} \Gamma^m_{kp} - \Gamma^p_{ki} \Gamma^m_{jp} \right) \\
&\quad + g_{lm} \left(\partial_i \Gamma^m_{kj} - \partial_k \Gamma^m_{ij} + \Gamma^p_{kj} \Gamma^m_{ip} - \Gamma^p_{ij} \Gamma^m_{kp} \right) \\
&= 0 \ ,
\end{aligned} \tag{3.28}
$$

since $\Gamma^k_{ij} = \Gamma^k_{ji}$.

This finishes the proof of the proposition. ■

Using the symmetries of the Riemann curvature tensor, we can show that the Ricci curvature tensor is a symmetric tensor.

Proposition 3.7. The Ricci curvature tensor is symmetric, i.e. $R_{jl} = R_{lj}$.

Proof. We calculate using the previous proposition the Ricci curvature tensor R_{lj}

$$
\begin{aligned}
R_{lj} &= g^{ik} R_{ilkj} = g^{ik} R_{kjil} = -g^{ik} R_{jkil} = g^{ik} \left(R_{ijkl} + R_{kijl} \right) \\
&= R_{jl} + \frac{1}{2} g^{ik} \left(R_{kijl} - R_{ikjl} \right) = R_{jl} + \frac{1}{2} \left(g^{ki} R_{kijl} - g^{ik} R_{ikjl} \right) \\
&= R_{jl} \ ,
\end{aligned} \tag{3.29}
$$

which completes the proof. ■

Using the Christoffel symbols we can give the geodesic equation. Solutions to this differential equation are geodesics, which are the "shortest paths" on general manifolds.

Geodesics and the Exponential map

Definition 3.8 (Geodesics)**.** A geodesic $\gamma_v(t) : I \to M$ is the solution to the coupled ordinary differential equation

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0 \tag{3.30}$$

with initial conditions $x(0) = p$ and $\dot{x}(0) = v$.

Existence and uniqueness of solutions to ordinary differential equations implies the following lemma.

Lemma 3.9 (Rescaling lemma). For every $p \in M$, every $v \in T_pM$ and every $c, t \in \mathbb{R}$

$$\gamma_v(ct) = \gamma_{cv}(t) \tag{3.31}$$

whenever either side is defined.

For a full proof see Lemma 5.18 in [Lee18]. We denote the time-one solution by the exponential map.

Definition 3.10 (Exponential map)**.** The exponential map $\exp_p(v)$ is the time-one solution of the Geodesics Equation (3.30) with initial conditions $x(0) = p$ and $\dot{x}(0) = v$.

The inverse function theorem implies the following proposition.

Proposition 3.11. The exponential map is a local diffeomorphism.

Proof. By the inverse function theorem, it is sufficient to show that the derivative $d\exp_p : T_0T_pM \cong T_pM \to T_pM$ is a linear isomorphism. To compute this derivative, we take a curve τ starting at $0 \in T_pM$ with initial velocity v, e.g. $\tau(t) = tv$, and compute the initial velocity of $\exp_p \circ \tau$. We get

$$d(\exp_p)_0(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp_p \circ \tau)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(tv) = \gamma_v(t) = v \ , \tag{3.32}$$

by the Rescaling lemma. We therefore conclude that the derivative $d\exp_p$ is the identity map, and that the exponential map is a local diffeomorphism. ■

We use the exponential map in the definition of normal coordinates.

Normal coordinates

Definition 3.12 (Normal coordinates)**.** We say that a neighbourhood U of $p \in M$ is *normal* if it is the image of a star-shaped domain $V \subseteq T_pM$ under the local diffeomorphism $\exp_p$.

Taking an orthonormal basis (b^i) of T_pM with respect to the metric tensor g_{ij}, there is a linear transformation $B : \mathbb{R}^n \to T_pM$ such that the standard orthonormal basis $(e^1, \ldots, e^n)$ is mapped to E. We say that the local coordinates $(x^1, \ldots, x^n)$ given by the chart $\varphi = B^{-1} \circ (\exp_p|_V)^{-1} : U \to \mathbb{R}^n$ are the *normal coordinates* centred at p.

Using the definitions, propositions and lemmata from above we can give the following proposition.

Proposition 3.13 (Normal coordinates)**.** The following hold about normal coordinates:

(a) There are normal coordinates.

(b) The normal coordinates are unique up to multiplication by a matrix $A \in O(n)$, which is constant in x.

(c) The normal coordinates of $p \in M$ are $(0, 0, \ldots, 0)$.

(d) The Riemannian metric $g_{ij}(p) = \delta_{ij}$.

(e) If $v = v^i \partial_i \in T_pM$, then the geodesic γ_v starting at p with initial velocity v, is given by $\gamma_v(t) = (tv^1, \ldots, tv^n)$.

(f) The Christoffel symbols vanish at p.

(g) All the first partial derivatives of g_{ij} vanish at p.

(h) At p, the Riemann curvature R_{ijkl} is given by

$$\frac{1}{2}\left(\partial_i\partial_l g_{jk} + \partial_j\partial_k g_{il} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik}\right) \ . \tag{3.33}$$

Proof. (a) The definition provides a construction for normal coordinates.

(b) Suppose (e^i) and $(\tilde{e}^i)$ are normal coordinates centred around p, then the images (b^i) of (e^i) under $B : \mathbb{R}^n \to T_pM$ and $(\tilde{b}^i)$ of $(\tilde{e}^i)$ under $\tilde{B} : \mathbb{R}^n \to T_pM$ form orthonormal bases of T_pM with respect to the metric tensor g_{ij}. Hence, (b^i) differs from $(\tilde{b}^i)$ by an isometric transformation, and hence $B \circ B^{-1}$ is an isometry, such that $e^i = A^i_j \tilde{e}^j$ for some orthogonal matrix $A \in O(n)$.

(c) This follows from the definition, since $\exp_p(0) = p$.

(d) The coordinates $(b^i) \subseteq T_pM$ are orthonormal with respect to g_{ij}. Hence, in the normal coordinates $g_{ij} = \delta_{ij}$.

(e) The exponential map $\exp_p$ is the time-1 flow of a geodesic $\gamma_v(t)$ with initial point p and initial velocity v. It follows that

$$\begin{aligned} \varphi(\gamma_v(t)) &= B^{-1} \circ (\exp_p|_V)^{-1} \circ \exp_p|_V(tv) \\ &= B^{-1}(tv) = (tv^1, \ldots, tv^n) \end{aligned} \tag{3.34}$$

in normal coordinates.

(f) By the previous part with the geodesic equation at $t = 0$, it follows that

$$\dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(p) = v^i v^j \Gamma^k_{ij}(p) = 0 \ . \tag{3.35}$$

In particular for $v = \partial_a + \partial_b$ it shows that

$$\Gamma^k_{ab} = 0 \tag{3.36}$$

for all $a, b, k \in \{1, \dots, n\}$. We conclude that all Christoffel symbols vanish at p in normal coordinates.

(g) By the metric compatibility of the Christoffel symbol

$$\Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} = \partial_k g_{ij} \ , \tag{3.37}$$

it follows, using the previous part, that at p in normal coordinates $\partial_k g_{ij} = 0$.

(h) By the previous two parts of this proposition, it follows that

$$\begin{aligned} R_{ijkl} &= g_{lm} \left(\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} + \Gamma^p_{ik} \Gamma^m_{jp} - \Gamma^p_{jk} \Gamma^m_{ip} \right) \\ &= \delta_{lm} \left(\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} \right) \\ &= \frac{\delta_{lm}}{2} \Bigg(\partial_j \left(g^{mq} \left(\partial_i g_{kq} + \partial_k g_{iq} - \partial_q g_{ik} \right) \right) \\ &\qquad - \partial_i \left(g^{mq} \left(\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk} \right) \right) \Bigg) \\ &= \frac{1}{2} \left(\partial_i \partial_l g_{jk} + \partial_j \partial_k g_{il} - \partial_i \partial_k g_{jl} - \partial_j \partial_l g_{ik} \right) \ . \end{aligned} \tag{3.38}$$

This finishes the proof for the proposition. ■

The next theorem gives an expansion of the metric tensor in normal coordinates.

Theorem 3.2. *The Taylor expansion of the Riemannian metric $g_{ij}(x)$ in normal coordinates centred at p is given by*

$$g_{ij}(x) = \delta_{ij}(p) - \frac{1}{3} R_{ikjl}(p) x^k x^l + \mathcal{O}(|x|^3) \ . \tag{3.39}$$

Proof. We have from Equation (3.20) that in normal coordinates

$$\partial_l \partial_k g_{ij}(0) = \partial_l \Gamma^m_{ik}(0) \delta_{mj} + \partial_l \Gamma^m_{jk}(0) \delta_{im} \ . \tag{3.40}$$

By the fact that geodesics through the origin are given by tv it follows that by differentiating the geodesic equation at time $t = 0$ in normal coordinates that

$$0 = \partial_l \Gamma^k_{ij}(tv) v^i v^j = \partial_l \Gamma^k_{ij}(0) v^i v^j v^l \ , \tag{3.41}$$

which is a homogenous polynomial of degree 3. Since the coefficient of the term $v^i v^j v^l$ is the permutation of $\partial_l \Gamma^k_{ij}(0)$, it follows that

$$\partial_l \Gamma^k_{ij}(0) + \partial_i \Gamma^k_{jl}(0) + \partial_j \Gamma^k_{li}(0) = 0 \ . \tag{3.42}$$

Then in normal coordinates

$$
\begin{aligned}
R_{iklj} + R_{ilkj} &= \delta_{jm}\left(\partial_k \Gamma^m_{il} - \partial_i \Gamma^m_{kl}\right) + \delta_{jm}\left(\partial_l \Gamma^m_{ik} - \partial_i \Gamma^m_{lk}\right) \\
&= \delta_{jm}\left(-3\partial_i \Gamma^m_{lk}\right) \ ,
\end{aligned}
\tag{3.43}
$$

such that by Equation (3.40)

$$
\begin{aligned}
\partial_l \partial_k g_{ij}(0) &= -\frac{1}{3}\left(R_{likj} + R_{lkij} + R_{ljki} + R_{lkji}\right) \\
&= -\frac{1}{3}\left(R_{ikjl} + R_{iljk}\right) \ .
\end{aligned}
\tag{3.44}
$$

Then it also follows that

$$
g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + \mathcal{O}(|x|^3) \ , \tag{3.45}
$$

by the Taylor expansion, which finishes the proof. ■

We can now also give the second derivative of the inverse g^{ij} and the absolute value of the determinant $|g| = |\det g_{ij}|$ in normal coordinates centred at p.

Corollary 3.14. The second derivative $\partial_p \partial_m g^{ij}$ in normal coordinates at p is given by

$$
\partial_p \partial_m g^{ij}(p) = \frac{1}{3} \delta^{ki} \delta^{lj}\left(R_{kplm} + R_{kmlp}\right) \ . \tag{3.46}
$$

Proof. Notice that by Leibniz rule

$$
\begin{aligned}
\partial_p \partial_m g^{ij}(p) &= \partial_p \partial_m (g^{ik} g_{kl} g^{lj})(p) \\
&= \partial_p \partial_m (g^{ik})(g_{kl} g^{lj})(p) + g^{ik} g^{lj} \partial_p \partial_m (g_{kl})(p) + g^{ik} g_{kl} \partial_p \partial_m (g^{lj})(p) \\
&= 2\partial_p \partial_m g^{ij}(p) + \delta^{ki} \delta^{lj} \partial_p \partial_m g_{kl}(p)
\end{aligned}
\tag{3.47}
$$

from which the statement follows. ■

Corollary 3.15. The second derivative $\partial_p \partial_m |g|$ in normal coordinates at p is given by

$$
\partial_p \partial_m |g|(p) = -\frac{1}{3} \delta^{ij}\left(R_{imjp} + R_{ipjm}\right) = -\frac{2}{3} R_{pm} \ . \tag{3.48}
$$

Proof. In normal coordinates at p the determinant $\det g = 1$, hence on a neighbourhood of p, the determinant is positive, and we can take derivatives there. Since $g_{ij} = \delta_{ij} - \frac{1}{3} R_{ipjm} x^p x^m + \mathcal{O}(|x|^3)$, it follows that

$$
\det g = 1 - \frac{1}{3} \delta^{ij} R_{ikjl} x^k x^l + \mathcal{O}(|x|^3) = 1 - \frac{1}{3} R_{kl} x^k x^l + \mathcal{O}(|x|^3) \ , \tag{3.49}
$$

by the definition of the determinant, and the Ricci curvature in normal coordinates. The statement now follows directly. ■

In dimensions two and three, it is possible to write the Riemann curvature tensor in terms of the scalar curvature and the Ricci curvature respectively, as the following proposition shows.

Proposition 3.16. In dimensions two and three the Riemann curvature R_{abcd} tensor can be written as

$$R_{abcd} = \frac{R}{2}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right) \ , \tag{3.50}$$

and

$$R_{abcd} = R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad} + \frac{R}{2}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right) \tag{3.51}$$

respectively. In normal coordinates centred at $p \in M$, these expressions reduce to

$$R_{abcd}(p) = \frac{R}{2}\left(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}\right) \ , \tag{3.52}$$

and

$$R_{abcd}(p) = R_{ac}\delta_{bd} + R_{bd}\delta_{ac} - R_{ad}\delta_{bc} - R_{bc}\delta_{ad} + \frac{R}{2}\left(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}\right) \tag{3.53}$$

respectively.

Proof. From Proposition 3.6 it follows that R_{ijkl} has $n^2(n^2-1)/12$ independent components. For details of this claim see [Lee18] Proposition 7.21.

The upshot of this result is that if $\dim M = 2$ that R_{ijkl} has one independent component. Notice furthermore that $g_{ad}g_{bc} - g_{ac}g_{bd}$ satisfies the conditions of Proposition 3.6, hence

$$R_{abcd} = f(R)\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right) \tag{3.54}$$

for some smooth function f of the scalar curvature. By contracting the Riemann curvature to the scalar curvature we notice that

$$R = g^{bd}g^{ac}R_{abcd} = g^{bd}g^{ac}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right)f(R) = (4-2)f(R) \tag{3.55}$$

hence $f(R) = R/2$.

Similarly, if $\dim M = 3$, then R_{ijkl} has only six independent components, exactly the same number of independent components as R_{jl} and g_{ik} have. Notice that

$$R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad} + \frac{R}{2}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right) \tag{3.56}$$

satisfies the conditions of Proposition 3.6. Notice furthermore that

$$\begin{aligned}
&\left(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad} + \frac{R}{2}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right)\right)g^{ac} \\
&= g^{ac}R_{ac}g_{bd} + 3R_{bd} - R_{ad}\delta^a_b - R_{bc}\delta^c_d + \frac{R}{2}\left(g_{bc}\delta^c_d - 3g_{bd}\right) \\
&= Rg_{bd} + 3R_{bd} - 2R_{bd} - Rg_{bd} = R_{bd} \ ,
\end{aligned} \tag{3.57}$$

hence the map

$$\begin{aligned} G :&\{\text{symmetric 2-tensors}\} \to \{\text{curvature tensors}\} \\ R_{bd} &\mapsto R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad} + \frac{R}{2}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right) \end{aligned} \tag{3.58}$$

is a right-inverse to the map

$$\begin{aligned} \mathrm{Tr}_g :&\{\text{curvature tensors}\} \to \{\text{symmetric 2-tensors}\} \\ R_{abcd} &\mapsto g^{ac}R_{abcd} = R_{bd}\ . \end{aligned} \tag{3.59}$$

Thus, G is injective and Tr_g is surjective. But, because the dimensions of the spaces are equal, we see that G and Tr_g are isomorphisms. ■

We use these results in the next section to reduce the parametrix of the Laplace-Beltrami operator.

3.4 Parametrix of the Laplace-Beltrami operator in normal coordinates

Combining the results from the two previous sections, then we can see that τ_{-3} from Equation (3.11) vanishes at p_0 in normal coordinates. This is because the derivative $\partial_{x_{k_1}} g^{ij}(p_0) = 0$, and because the correction to τ_{-3} from Equation (3.13) vanishes also at p_0 in normal coordinates, since $\sigma_1(p_0,\xi)$ is a sum of derivatives of g^{ij} at p_0. Furthermore, we are able to write τ_{-4} as

$$\begin{aligned} \tau_{-4}(p_0,\xi) = &\sum_{k_2=1}^{n}\sum_{k_3=k_2+1}^{n} -\left\langle \xi, \partial_{x_{k_2}}\partial_{x_{k_3}} g^{ij}\xi \right\rangle \left(\frac{-8\xi_{k_3}\xi_{k_2}|\xi|^2}{|\xi|^{10}}\right) \\ &+ \sum_{k_2=1}^{n} \frac{-1}{2}\left\langle \xi, \partial_{x_{k_2}}\partial_{x_{k_2}} g^{ij}\xi \right\rangle \left(\frac{2}{|\xi|^6} - \frac{8|\xi|^2\xi_{k_2}^2}{|\xi|^{10}}\right) \\ &+ \sum_{i,k,j=1}^{n} \left(\frac{1}{2}\left(\partial_{x_k}\partial_{x_i}|g|\right)\delta^{ij}\xi_j + (\partial_{x_k}\partial_{x_i} g^{ij})\xi_j\right)\frac{2\xi_k}{|\xi|^6} \end{aligned} \tag{3.60}$$

In the rest of this section we calculate this term in the case that the dimension n is two and three.

Dimension two

Lemma 3.17. In normal coordinates in dimension 2 we can write

$$\partial_{x_p}\partial_{x_m}\sigma_2(p_0,\xi) = \frac{R}{3}\left(\delta_{pm}|\xi|^2 - \xi_m\xi_p\right) \tag{3.61}$$

Proof. Using Corollary 3.14 and Proposition 3.16 we see that

$$\begin{aligned}\partial_{x_m}\partial_{x_p}\sigma_2(p_0,\xi) &= -\partial_{x_m}\partial_{x_p}g^{ij}\xi_i\xi_j \\ &= -\frac{1}{3}(R_{kplm}+R_{kmlp})\xi^k\xi^l \\ &= -\frac{R}{6}\left(\delta_{km}\delta_{pl}-\delta_{kl}\delta_{pm}+\delta_{kp}\delta_{ml}-\delta_{kl}\delta_{mp}\right)\xi^k\xi^l \\ &= \frac{R}{6}\left(2|\xi|^2\delta_{pm}-2\xi_m\xi_p\right)\end{aligned} \tag{3.62}$$

which proves Equation (3.61). ∎

Lemma 3.18. In normal coordinates in dimension 2 we can write

$$\partial_{x_p}\sigma_1(p_0,\xi) = -i\frac{R}{3}\xi_p \tag{3.63}$$

Proof. Once again using Corollary 3.14 and Proposition 3.16, but also by Corollary 3.15 we find that

$$\begin{aligned}\partial_{x_p}\sigma_1(p_0,\xi) &= -i\left(\frac{\partial_{x_p}\partial_{x_i}|g|}{2|g|}g^{ij}+\partial_{x_p}\partial_{x_i}g^{ij}\right)\xi_j \\ &= -i\left(-\frac{1}{6}\left(R_{kikp}+R_{kpki}\right)\delta^{ij}+\frac{1}{3}\delta^{ki}\delta^{lj}\left(R_{kpli}+R_{kilp}\right)\right)\xi_j \\ &= -i\Bigg(-\frac{R}{12}\left(\delta_{kp}\delta_{ik}-\delta_{kk}\delta_{ip}+\delta_{ki}\delta_{kp}-\delta_{kk}\delta_{pi}\right)\delta^{ij} \\ &\qquad +\frac{R}{6}\delta^{ki}\delta^{lj}\left(\delta_{ki}\delta_{pl}-\delta_{kl}\delta_{pi}+\delta_{kp}\delta_{li}-\delta_{kl}\delta_{ip}\right)\Bigg)\xi_j \\ &= -i\left(\frac{R}{6}\xi_p+\frac{R}{6}\xi_p\right)\end{aligned} \tag{3.64}$$

which proves Equation (3.63). ∎

Proposition 3.19. In dimension 2, the -4^{th} order term of the parametrix of the symbol of the Laplace-Beltrami operator in normal coordinates centred at p_0 is given by

$$\tau_{-4}(p_0,\xi) = -\frac{R}{|\xi|^4}\ . \tag{3.65}$$

Proof. Using the previous two lemmata, we see that

$$\begin{aligned}\tau_{-4}(p_0,\xi) = -\Bigg[&-\sum_{1\le p<m\le 2}\frac{R}{3}\left(\delta_{pm}|\xi|^2-\xi_m\xi_p\right)\left(\frac{-8\xi_p\xi_m}{|\xi|^6}\right) \\ &-\sum_{1\le p=m\le 2}\frac{R}{6}\left(\delta_{pm}|\xi|^2-\xi_m\xi_p\right)\left(\frac{2\delta_{pm}}{|\xi|^4}-\frac{8\xi_p\xi_m}{|\xi|^6}\right)-i\sum_{1\le p\le 2}-i\frac{R}{3}\xi_p\frac{2\xi_p}{|\xi|^4}\Bigg]\tau_{-2} \\ = -\frac{8}{3}\frac{R\xi_1^2\xi_2^2}{|\xi|^8}&-\frac{R}{3}\left(\frac{\xi_1^2+\xi_2^2}{|\xi|^6}-\frac{8\xi_1^2\xi_2^2}{|\xi|^8}\right)-\sum_p\frac{2R\xi_p^2}{3|\xi|^6} = -\frac{R}{|\xi|^4}\ .\end{aligned} \tag{3.66}$$

This proves the proposition. ∎

Dimension three

In three dimensions the calculations become a little more elaborate.

Lemma 3.20. In normal coordinates in dimension 3 we can write

$$\begin{aligned} \partial_{x_m}\partial_{x_p}\sigma_2(p_0,\xi) = -\frac{1}{3}\Big[& 2R_{kl}\delta_{pm}\xi^k\xi^l + 2R_{pm}|\xi|^2 - 2R_{km}\xi^k\xi_p \\ & - 2R_{lp}\xi^l\xi_m + R\left(\xi_m\xi_p - \delta_{pm}|\xi|^2\right)\Big] \end{aligned} \tag{3.67}$$

Proof. Using Corollary 3.14 and Proposition 3.16 we see that

$$\begin{aligned} & \partial_{x_m}\partial_{x_p}\sigma_2(p_0,\xi) = -\partial_{x_m}\partial_{x_p}g^{ij}\xi_i\xi_j = \partial_{x_m}\partial_{x_p}g_{ij}\xi_i\xi_j \\ & = -\frac{1}{3}(R_{kplm} + R_{kmlp})\xi^k\xi^l \\ & = -\frac{1}{3}\Big[2R_{kl}\delta_{pm}\xi^k\xi^l + 2R_{pm}|\xi|^2 - R_{km}\xi^k\xi_p - R_{lp}\xi^l\xi_m \\ & \quad - R_{kp}\xi^k\xi_m - R_{lm}\xi^l\xi_p + \frac{R}{2}\left(2\xi_m\xi_p - 2\delta_{pm}|\xi|^2\right)\Big] \end{aligned} \tag{3.68}$$

which proves the lemma after relabelling the necessary indices. ■

Lemma 3.21. In normal coordinates in dimension 3 we can write

$$\partial_{x_p}\sigma_1(p_0,\xi) = \frac{5i}{3}R_{ip}\xi^i \ . \tag{3.69}$$

Proof. Using Corollaries 3.14 and 3.15 we find that

$$\begin{aligned} & \partial_{x_p}\sigma_1(p_0,\xi) = -i\left(\frac{\partial_{x_p}\partial_{x_i}|g|}{2|g|}g^{ij} + \partial_{x_p}\partial_{x_i}g^{ij}\right)\xi_j \\ & = -i\left(-\frac{1}{6}\left(R_{kikp} + R_{kpki}\right)\delta^{ij} + \frac{1}{3}\delta^{ki}\delta^{lj}\left(R_{kpli} + R_{kilp}\right)\right)\xi_j \ , \end{aligned} \tag{3.70}$$

which, using Proposition 3.16, and the fact that the trace of the Ricci curvature is the scalar curvature, can be reduced to

$$\begin{aligned} & \frac{i}{3}\Bigg[\left(R_{kk}\delta_{pi} + R_{pi}\delta_{kk} - R_{kp}\delta_{ik} - R_{ik}\delta_{kp} + \frac{R}{2}\left(\delta_{kp}\delta_{ki} - \delta_{kk}\delta_{ip}\right)\right)\delta^{ij} \\ & - \delta^{ki}\delta^{lj}\Big[\left(2R_{kl}\delta_{ip} + 2R_{ip}\delta_{kl} - R_{ki}\delta_{pl} - R_{pl}\delta_{ik} - R_{kp}\delta_{il} - R_{il}\delta_{kp}\right) \\ & \quad + \frac{R}{2}\left(\delta_{ki}\delta_{pl} - 2\delta_{kl}\delta_{pi} + \delta_{kp}\delta_{il}\right)\Big]\Bigg]\xi_j \\ & = \frac{i}{3}\left(5R_{ip}\xi^i + R\xi_p - \frac{R}{2}\left(3\xi_p - 2\xi_p + \xi_p\right)\right) = \frac{5i}{3}R_{ip}\xi^i \ , \end{aligned} \tag{3.71}$$

which proves the lemma. ■

Proposition 3.22. In dimension 3, the -4^{th} order term of the parametrix of the symbol of the Laplace-Beltrami operator in normal coordinates centred at p_0 is given by

$$\tau_{-4}(p_0,\xi) = \frac{32}{3}\frac{R_{ip}\xi_i\xi_p}{|\xi|^6} \ . \tag{3.72}$$

Proof. Using the previous two lemmata, computeralgebra systems can find that

$$\begin{aligned}\tau_{-4}(p_0,\xi) = \frac{1}{|\xi|^2}\Bigg[\\ \sum_{1\le p<m\le 3} \frac{1}{3}\left[2R_{pm}|\xi|^2 - 2R_{km}\xi^k\xi_p - 2R_{lp}\xi^l\xi_m + R\xi_m\xi_p\right]\left(\frac{-8\xi_p\xi_m}{|\xi|^6}\right) \\ + \sum_p \Bigg\{ \frac{1}{6}\left[2R_{kl}\delta_{pp}\xi^k\xi^l + 2R_{pp}|\xi|^2 - 4R_{kp}\xi^k\xi_p + R\left(\xi_p^2 - \delta_{pp}|\xi|^2\right)\right] \\ \left(\frac{2\delta_{pp}}{|\xi|^4} - \frac{8\xi_p^2}{|\xi|^6}\right)\Bigg\} + \sum_p \frac{1}{3}\left(5R_{ip}\xi^i\right)\frac{2\xi_p}{|\xi|^4}\Bigg] = \frac{32}{3}\frac{R_{ij}\xi^i\xi^j}{|\xi|^6}\end{aligned} \tag{3.73}$$

which finishes the proof. ■

3.5 Schwartz kernels

In this section we look at the Schwartz kernels of the parametrices we found in the previous section. Remember Theorem 2.5, which shows that the Schwartz kernel exists. This result shows that we are allowed to calculate the Schwartz kernels $K_{-2-l}(p_0,p)$ for $l = 0,1,2,\dots$.

In this section we calculate the Schwartz kernels for the leading two terms of the parametrix of the Laplace-Beltrami operator in dimension two and three.

Theorem 3.3. *The Schwartz kernel of order* -2 *is given by*

$$K_{-2}(p_0,p) = \begin{cases} \log|p_0-p| + K'_{-2}(p_0,p) & n=2 \\ 2^{(n-4)/2}\Gamma(\frac{n}{2}-1)|p-p_0|^{2-n} + K'_{-2}(p_0,p) & n\ge 3\end{cases} \tag{3.74}$$

where $K'_{-2}(p_0,p)$ *is infinitely smoothing.*

Proof. Notice first that if $a(x,\xi)\in\mathcal{S}^k$ is a symbol, which has compact support in the ξ-variable as in the assumptions of Corollary 2.7, then the Schwartz kernel $K_a(x,y)$ is also infinitely smoothing. This is because

$$(T_a u)(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}a(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \in C^\infty(\mathbb{R}^n) \ , \tag{3.75}$$

however at the same time

$$(T_a u)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} K_a(x,y) u(y) \, \mathrm{d}y \; . \tag{3.76}$$

Now the theorem follows directly from Theorem 1.3. ■

Theorem 3.4. *On $(\mathbb{R}^2, g)$, the Schwartz kernel K_{-4} in normal coordinates centred at p_0 is given by*

$$\begin{aligned} K_{-4}(p_0, p) = &-\frac{1}{2} R(p_0) |p - p_0|^2 \\ &+ \frac{1}{2} R(p_0) |p - p_0|^2 \log |p - p_0| + K'_{-4}(p_0, p) \; , \end{aligned} \tag{3.77}$$

with $K'_{-4}(p_0, p)$ infinitely smoothing.

Proof. Notice that since $R/|\xi|^4$ is rotationally invariant

$$\mathcal{F}\left[\frac{R}{|\xi|^4}\right](x) = \mathcal{F}^{-1}\left[\frac{R}{|\xi|^4}\right](-x) = \mathcal{F}^{-1}\left[\frac{R}{|\xi|^4}\right](x) \; , \tag{3.78}$$

hence by Propositions 1.27 and 3.19, the theorem holds. ■

Theorem 3.5. *On $(\mathbb{R}^3, g)$ the Schwartz kernel K_{-4} in normal coordinates centred at p_0 is given by*

$$\begin{aligned} K_{-4}(p_0, p) = \frac{2\sqrt{2\pi}}{3} \Bigg(& R(p_0) |p - p_0| \\ &+ \frac{R_{ij}(p_0)(p - p_0)^i (p - p_0)^j}{|p - p_0|} \Bigg) + K'_{-4}(p_0, p) \end{aligned} \tag{3.79}$$

with K'_{-4} infinitely smoothing.

Proof. By Proposition 1.28 it follows in normal coordinates centred at p_0 that

$$\begin{aligned} \mathcal{F}^{-1}\left[\frac{R_{ij}(p_0)\xi^i \xi^j}{|\xi|^6}\right](y) &= \mathcal{F}\left[\frac{R_{ij}(p_0)\xi^i \xi^j}{|\xi|^6}\right](-y) \\ &= \frac{\sqrt{2\pi}}{16} R_{ij}(p_0) \left(\delta^{ij} |y| + \frac{y^i y^j}{|y|}\right) \\ &= \frac{\sqrt{2\pi}}{16} \left(R(p_0)|y| + \frac{R_{ij}(p_0) y^i y^j}{|y|}\right) \; . \end{aligned} \tag{3.80}$$

Hence, on $(\mathbb{R}^3, g)$ in normal coordinates centred at p_0, it holds that

$$\begin{aligned} K_{-4}(x,y) &= \frac{32}{3} \mathcal{F}^{-1}\left[\frac{R_{ij}(p_0)\xi^i \xi^j}{|\xi|^6}\right] + K'_{-4}(x,y) \\ &= \frac{2\sqrt{2\pi}}{3} \left(R(p_0)|y| + \frac{R_{ij}(p_0) y^i y^j}{|y|}\right) + K'_{-4}(x,y) \; , \end{aligned} \tag{3.81}$$

with $K'_{-4}(x,y)$ infinitely smoothing, by Proposition 3.22, which proves the theorem. ■

The following theorem allows us to calculate the solutions to the pseudodifferential equation

$$\Delta_g u = f \ . \tag{3.82}$$

Theorem 3.6. *Let $n = 2$ or $n = 3$, then for each $f \in L^2$, the function u' given by*

$$u' = \int_{\mathbb{R}^n} K_{-2}(x, x-z) f(z) \, \mathrm{d}z + \int_{\mathbb{R}^n} K_{-4}(x, x-z) f(z) \, \mathrm{d}z \tag{3.83}$$

satisfies

$$u - u' \in \begin{cases} C^2 & \text{if } n = 2 \\ C^1 & \text{if } n = 3 \end{cases} , \tag{3.84}$$

where u denotes the solution to the pseudodifferential equation (3.82).

Proof. Using the parametrix P of the Laplace-Beltrami operator Δ_g we find a u' such that for each f it holds

$$u - Pf \in C^\infty(\mathbb{R}^n) \tag{3.85}$$

for each n. Using the calculations above, we can write Pf as

$$Pf(x) = \int_{\mathbb{R}^n} (K_{-2}(x, x-z) + K_{-2}(x, x-z)) f(z) \, \mathrm{d}z + (T_{\tau_{-5}} f)(x) \ . \tag{3.86}$$

We only need to show that $T_{\tau_{-5}} f$ is sufficiently continuously differentiable. Let $g \in L^2$ be such that $\bar{g} = \hat{f}$, then by the Cauchy-Schwartz inequality it follows that

$$\begin{aligned} |\partial_x^\alpha (T_{\tau_{-5}} f)(x)| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\xi^\alpha \tau_{-5}(x, \xi) \hat{f}(\xi)| \, \mathrm{d}\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\xi^\alpha \tau_{-5}(x, \xi) \bar{g}(\xi)| \, \mathrm{d}\xi \\ &\leq (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} |C(1 + |\xi|)^{2(|\alpha| - 5)} \, \mathrm{d}\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} |g(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} , \end{aligned} \tag{3.87}$$

since $\xi^\alpha \tau_{-5}(x, \xi)$ is a symbol of order $|\alpha| - 5$. By using spherical coordinates we find that

$$\int_{\mathbb{R}^n} (1 + |\xi|)^{2(|\alpha| - 5)} \, \mathrm{d}\xi < \infty \tag{3.88}$$

if and only if $|\alpha| < 1 + (5 - n)/2$, which means in the case that $n = 2$ that $|\alpha| \leq 2$ and in the case that $n = 3$ that $|\alpha| \leq 1$. ■

This theorem shows that given $f \in L^2$ we can find an approximation u' given by Equation (3.83) such that u' differs from u by at most a C^1 function in the case that the dimension $n = 3$ and by at most a C^2 function in the case that the dimension $n = 2$.

3.6 Final remarks

This section will try to give some small insights into the use of this thesis.

Firstly, the Laplace-Beltrami operator: the operator can be given independently of a choice of local coordinates. In particular, on a general Riemannian manifold (M, g), we have $\Delta_g f = \operatorname{div}(\operatorname{grad} f)$ for any $f \in C^\infty(M, \mathbb{R})$. Notice that in this case this definition for the Laplace-Beltrami operator is consistent with the Laplace operator on $M = \mathbb{R}^n$ and $g_{ij} = \delta_{ij}$.

In an equivalent way to the Laplace operator, the Laplace-Beltrami operator can be used in partial differential equations on Riemannian manifolds. We already discussed the Poisson equation

$$\Delta_g u = f \tag{3.89}$$

in this thesis, but the heat-equation

$$\frac{\partial u(x,t)}{\partial t} = k \Delta_g u(x,t) \qquad k > 0 \ , \tag{3.90}$$

and wave-equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \Delta_g u(x,t) \qquad c \in \mathbb{R} \tag{3.91}$$

extend naturally to Riemannian manifolds in this way; see for example [DP20; CS19; Tzo23] and many others. Furthermore, some stochastic processes, which can be modelled on $\mathbb{R}^n$ by partial differential equations can now be modelled on Riemannian manifolds. The most important one is the random walk, which can be modelled by the heat equation, also on Riemannian manifolds; see for example [NTT21].

In general on a Riemannian manifold, there may not be local coordinates, which extend globally. Roughly, to use the calculations in this thesis, we give normal local coordinates x^i centred at a point p_0, and extend these local coordinates to $\mathbb{R}^n$ by defining a metric $\tilde{g}_{ij}$ which is g_{ij} inside some ball of g-radius $\varepsilon_0 > 0$ and δ_{ij} outside some ball of g-radius $\varepsilon_1 > \varepsilon_0$ around p_0. This can be achieved in a way such that $\tilde{g}_{ij}$ is still smooth. Once such coordinates are given, we are in the situation of the assumptions of the calculations in this thesis.

The calculations in this thesis can therefore be used for finding approximate solutions of the Poisson equation on general Riemannian manifolds.

Conclusion

In this thesis the theory of pseudodifferential operators was explored. In Chapter 1 we first gave an introduction to distribution theory, and the Fourier transform. We showed that Schwartz functions approximate the tempered distributions. We then showed that the Fourier transform extends to a linear isometric isomorphism on L^2. Finally, we gave a method for computing the Fourier transform of homogeneous distributions, and computed the Fourier transform of the distributions $|x|^{-2}$, $|x|^{-4}$ and others in the two- and three-dimensional cases.

In Chapter 2 we defined pseudodifferential operator using the theory built up in the previous chapter. We showed that the composition of two pseudodifferential operators is once again a pseudodifferential operator, and gave the symbol accompanying this operator. We then showed that a pseudodifferential operator of order 0 is a bounded linear operator on L^2, and used it to show that any pseudodifferential operator is a bounded linear operator between Sobolev spaces. Furthermore, we showed that a pseudodifferential operator has a Schwartz kernel, which allows us to represent the pseudodifferential operator as an integral operator.

Next we restricted ourselves to elliptic pseudodifferential operators, for which we showed that an approximate inverse, the parametrix exists. We used this parametrix to give a statement of elliptic regularity in Theorem 2.7.

Finally, in Chapter 3 we used all the material we discussed in the previous chapters to compute the parametrix of the Laplace-Beltrami operator Δ_g. We showed that normal coordinates exist on a Riemannian manifold to simplify the expression we found. Furthermore, we computed the Schwartz kernel of the terms in the parametrix of the Laplace-Beltrami operator, using the computed Fourier transforms of homogeneous distributions in Chapter 1. To conclude, we showed that if u' is of the form of Equation (3.83) and u is the solution to Equation (3.82), then $u - u'$ is C^1 if the dimension is equal three and $u - u'$ is even C^2 if the dimension is equal two. This shows that given an input function f we can compute the solution to Equation (3.82) up to a function which is continuously differentiable if the dimension is small enough.

To finish, we gave some very rough outlines of how to use the calculations in this thesis. I hope to be able to continue my research, use this material in further work and get a deeper understanding of the material covered in this thesis.

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Popular summary

In this thesis we explore partial differential equations on manifolds. We start by introducing constant coefficient partial differential operators $\mathbf{D}$. An example of these is the Laplacian given by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \ ,$$

which is the sum of the second derivative in each spatial direction of a function $u : \mathbb{R}^n \to \mathbb{R}$. For any $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x < \infty \ ,$$

we can easily solve the partial differential equation

$$\Delta u(x) = f(x) \qquad x \in \mathbb{R}^n \ ,$$

using the Fourier transform. In particular the solution is given by

$$u = \mathcal{F}^{-1}[-\hat{f}(\xi)/|\xi|^2] \ ,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and $\hat{f}(\xi)$ denotes the Fourier transform of $f(x)$. Unfortunately once the partial differential operator $\mathbf{D}$ is also dependent on the position x, this solution breaks down.

This thesis gives an introduction to so-called *pseudodifferential operators*, which extend the notion of space-dependent partial differential operators. We show that two pseudodifferential operators can be composed to a new pseudodifferential operator. Furthermore, we show that if a pseudodifferential operator T_a is *elliptic*, then an approximate inverse T_b can be found, allowing us to solve the pseudodifferential equation

$$T_a u = f \ ,$$

where f is as before. In particular, we find that $u = T_b f + g$, where g is an undetermined smooth function.

The theory of pseudodifferential operators is necessary, since on a smooth manifold M we can only give a local description of the spatial directions. We therefore cannot expect partial differential operators with constant spatial coefficients. An example of a pseudodifferential operator on a smooth manifold is given by the *Laplace-Beltrami* operator Δ_g on a Riemannian manifold (M, g). The Laplace-Beltrami operator extends the notion of the Laplacian to Riemannian manifolds, but is space-dependent, and therefore requires the theory of pseudodifferential operators. This thesis computes the leading order terms (up to 3^{rd} order) T of the approximate inverse of the Laplace-Beltrami operator, and shows that the solution u to the pseudodifferential equation

$$\Delta_g u = f \ ,$$

is given by $u = Tf + h$, where h is a twice continuously differentiable function if the dimension $n = 2$.

The Laplace-Beltrami operator can be used to define *random walks* on Riemannian manifolds in a similar fashion as to how the Laplacian can be used to define random walks on flat Euclidean space. This thesis does not go into that topic, but it indicates a possible use for the Laplace-Beltrami operator.