Mean first escape time on Asymptotically hyperbolic surfaces

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Let (M, g) be a Riemannian manifold, we analyse Brownian motion on M

- Narrow escape problems
  - Mean first escape times (MFET) of random walks
- Semi-formally: *Given a Brownian motion* X_t *starting at* $X_0 = x$ *and a target* Γ_ϵ, *what is the mean first escape time of the Brownian motion* X_t *into* Γ_ϵ, *and what are the asymptotics as* $\epsilon \to 0$*?*

Has a rich literature based on probabilistic, PDE, matched asymptotic and microlocal techniques.

Backgroud II

- Given a Riemannian manifold (M, g), Brownian motion is a continuous random variable $X_t, t \geq 0$.
- If X_t starts at $x_0 \in M$ (i.e. $X_0 = x_0$), then the probability density function $f(t,x)$ of X_t is given by the fundamental solution of the heat equation

$$\frac{\partial f(t,x)}{\partial t} = \Delta_g f(t,x) \ , \qquad f(0,x) = \delta_{x_0}(x) \ . \tag{1}$$

- Given a *trap* $\Gamma_\epsilon \subset M/\partial M$, denote the first escape time τ_ϵ of a Brownian motion X_t into Γ_ϵ

$$\tau_\epsilon = \inf\left\{t \geq 0 : X_t \in \Gamma_\epsilon\right\} \ .$$

- Interested in the mean first escape time $u_\epsilon(x)$

$$u_\epsilon(x) = \mathbb{E}[\tau_\epsilon : X_0 = x] \tag{2}$$

and their asymptotics as $\epsilon \to 0$.

Introduction & Background III

- Interested in the mean first escape time $u_\epsilon(x)$

$$u_\epsilon(x) = \mathbb{E}[\tau_\epsilon : X_0 = x] \tag{2}$$

and their asymptotics as $\epsilon \to 0$.

- Integration by parts argument shows

$$\Delta_g u_\epsilon(x) = -1 \quad \text{on } M \setminus \Gamma_\epsilon, \quad u_\epsilon|_{\partial\Gamma_\epsilon} = 0 \tag{3}$$

- Additional conditions can be added e.g. (partially) reflecting boundaries.

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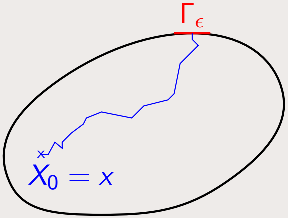
Narrow escape in 2D (Holcman & Schuss, 2004)

Let $\Omega \subset \mathbb{R}^2$ bounded with smooth boundary. The mean first escape time $u_\epsilon(x)$ of a Brownian motion (BM) starting at x through a small absorbing trap $\Gamma_\epsilon \subset \partial\Omega$ on an otherwise smooth completely reflecting boundary of radius ϵ satisfies

$$u_\epsilon = -|\Omega| \log \epsilon + \mathcal{O}(1) \quad \text{as } \epsilon \to 0.$$



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Narrow escape in 3D Riemannian manifolds (Nursultanov, Tzou & Tzou 2021)

$(M^3, \partial M, g)$ a smooth compact Riemannian 3-manifold with boundary. The mean first escape time $u_\epsilon(x)$ of a BM X_t starting at x through a small absorbing trap $\Gamma_\epsilon \subset \partial M$ on an otherwise smooth completely reflecting boundary of radius ϵ satisfies

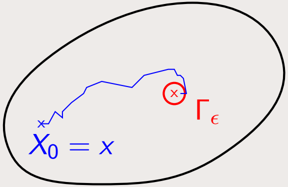
$$u_\epsilon = \frac{|M|}{4\epsilon} - \frac{1}{4\pi} H(x^*)|M| \log \epsilon + \mathcal{O}(1)$$

where $H(x^*)$ is the mean curvature at the centre of the trap.

Narrow capture on general Riemannian surfaces (Nursultanov, Trad, Tzou & Tzou 2023)

Let (M, g) be a smooth compact Riemannian manifold (w, w/o completely reflecting boundary ∂M) of dimension 2. The mean first escape time $u_\epsilon(x)$ of a Brownian motion starting at x into a geodesic ball $\Gamma_\epsilon = B_{x_0}(\epsilon)$ of size ϵ around $x_0 \in M^\circ$ satisfies

$$u_\epsilon(x) = -\frac{|M|_g}{2\pi} \log \epsilon + \mathcal{O}(1) \quad \text{as } \epsilon \to 0.$$



Background V

Asymptotically hyperbolic & gas giant manifolds

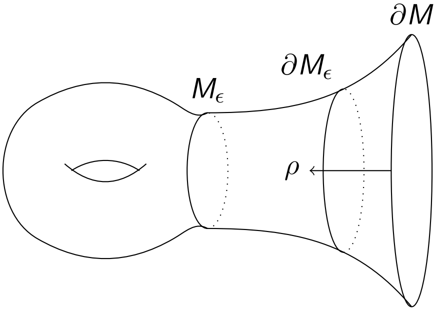
Asymptotically hyperbolic manifolds

- Riemannian manifolds with boundary $(M, \partial M, g)$ where $g = \bar{g}/\rho^2$ for some boundary defining function ρ, and some compact metric $\bar{g}$.
  - Natural extension the hyperbolic spaces $\mathbb{H}^n$
  - Negatively curved (near the boundary ∂M)
  - Infinite length geodesics and infinite volume

Gas giant manifolds

- Riemannian manifolds with boundary $(M, \partial M, g)$ where $g = \bar{g}/\rho^\alpha$ for some boundary defining function ρ, some compact metric $\bar{g}$ and some parameter $\alpha \in (0, 2)$.
  - Introduced recently
  - Finite length geodesics, but (depending on the dimension and parameter α) infinite volume

Setting the setting



- Normalise $|\mathrm{d}\rho|_{\partial M} = 1 > 0$
- Let $\Gamma_\epsilon = \{x \in M : \rho(x) \leq \epsilon\}$, $M_\epsilon := M \setminus \Gamma_\epsilon = \{x \in M : \rho(x) > \epsilon\}$.
- Mean first escape time $u_\epsilon(x)$ of a Brownian motion X_t starting at $x \in M_\epsilon$ into Γ_ϵ satisfies the PDE

$$\Delta_g u(x) = -1 \quad \text{on } M_\epsilon, \quad u_\epsilon|_{\partial M_\epsilon} = 0 \tag{4}$$

Theorems

MFET of BM on AH and gas giant surfaces

Theorem (J. Gell-Redman, EJG, J. Tzou, L. Tzou)

If $(M, \partial M, g)$ is an asymptotically hyperbolic or gas giant surface, then the mean first escape time from the manifold $M_\epsilon = \{x \in M : \rho(x) \geq \epsilon\}$ satisfies

$$u_\epsilon(x) = \begin{cases} -\log \epsilon + \mathcal{O}(1) & \textit{if } M \textit{ is asymptotically hyperbolic} \\ \mathcal{O}(1) & \textit{if } M \textit{ is a gas giant} \end{cases} \quad \textit{as } \epsilon \to 0.$$

Strategy I

Lemma

Let M be a two dimensional Riemannian manifold with two metrics g_1, g_2 conformally related by $g_2 = \beta(x) g_1$ for some positive smooth function $\beta(x)$, then

$$\Delta_{g_2} = \Delta_{\beta(x) g_1} = \beta(x) \Delta_{g_1}$$

Corollary

Let $(M, g = \bar{g}/\rho^2)$ be asymptotically hyperbolic, then the boundary value problem (4) is equivalent to

$$\Delta_{\bar{g}} u_\epsilon(x) = -\frac{1}{\rho^2(x)} \quad on \ M_\epsilon, \quad u_\epsilon|_{\partial M_\epsilon} = 0 \ .$$

Strategy II

Let $v_\epsilon(x) \in C^\infty(M_\epsilon)$ be defined by

$$v_\epsilon(x) := \log \rho(x) - \log \epsilon \tag{5}$$

Lemma

If $(M = \mathbb{D}, g = 4\overline{g}_{\mathrm{Euc}}/(1 - x^2 - y^2)^2)$ is the (constantly negatively curved) Poincaré disc with boundary defining function $\rho(x) = \frac{1}{2}(1 - x^2 - y^2)$, then $v_\epsilon(x)$ solves the boundary value problem (4), i.e. satisfies the bound $u_\epsilon(x) = -\log \epsilon + \mathcal{O}(1)$ as $\epsilon \to 0$.

Strategy

On general (M, g) asymptotically hyperbolic, near the boundary $v_\epsilon(x)$ solves the boundary value problem (4) (approximately). Take a bump-function $\chi(\rho(x)) \equiv 1$ near the boundary ∂M. If $u_\epsilon(x)$ solves the boundary value problem (4), set

$$w_\epsilon(x) := u_\epsilon(x) - \chi(x) v_\epsilon(x) \tag{6}$$

and show asymptotics of $w_\epsilon(x)$.

Proving strategy II

Proposition

The function w_ϵ satisfies

$$\Delta_{\bar{g}} w_\epsilon(x) = f_1(x)\log\epsilon + f_2(x) \quad \textit{for } x \in M_\epsilon$$

with

- $f_1, f_2 \in C^\infty(M_\epsilon)$
- *If $G_0(x;y)$ is the Dirichlet Green's function for $\Delta_{\bar{g}}$ on $\overline{M}$, then*

$$\int_{M_\epsilon} G_0(x;y) f_1(y) \mathrm{dVol}_{\bar{g}}(y) = \chi(x) - 1$$

- *We have for each $x \in M^\circ$*

$$\int_{M_\epsilon} G_0(x;y) f_2(y) \mathrm{dVol}_{\bar{g}}(y) = \mathcal{O}(1) \quad \textit{as } \epsilon \to 0.$$

Layer potentials

Definition

For $f \in C^\infty(\partial\Omega)$, the single layer potential is the operator

$$\mathrm{SL} : C^\infty(\partial\Omega) \to C^\infty(\Omega) : f \mapsto \int_{\partial\Omega} G(x;y)f(y)\mathrm{d}S(y)$$

Proposition

For $f \in C^\infty(\partial\Omega)$, the single layer potential satisfies the jump relations

$$\lim_{x\to z^+} \mathrm{SL}f(x) = \lim_{x\to z^-} \mathrm{SL}f(x) = \int_{\partial\Omega} G(z;y)f(y)\mathrm{d}S(y)$$

$$\lim_{x\to z^\pm} \partial_\nu \mathrm{SL}f(x) = \frac{1}{2}\left(\mp f(z) + 2\int_{\partial\Omega} \partial_{\nu_z} G(z;y)f(y)\mathrm{d}S(y)\right) = \frac{1}{2}(\mp I + N^\#)f(z)$$

Layer potentials II

By Green's formula we have

$$w_\epsilon(x) = \int_{M_\epsilon} G_0(x;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y) - \int_{\partial M_\epsilon} G_0(x;y)\partial_\nu w_\epsilon(y)\mathrm{d}S(y)$$

Goal: Find asymptotics of $\partial_\nu w_\epsilon(y)$ as $\epsilon \to 0$.

Apply ∂_ν:

$$\partial_\nu w_\epsilon(z) + \partial_\nu \int_{\partial M_\epsilon} G_0(x;y)\partial_\nu w_\epsilon(y)\mathrm{d}S(y) = \partial_\nu \int_{M_\epsilon} G_0(x;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y)$$

By layer potentials, get the equation

$$\frac{1}{2}(I + N_\epsilon^{\#})(\partial_\nu w_\epsilon)(z) = \partial_\nu \int_{M_\epsilon} G_0(z;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y)$$

Lemma (Informally)

If $\Omega = \mathbb{D}_{\rho \geq \epsilon} \subset \mathbb{D}$, *and* $G_0(x; y)$ *is the Dirichlet Green's function on* $(\mathbb{D}, g_{\mathrm{Euc}})$, *then*

$$\partial_{\nu_z} G_0(z; y)|_{y \in \partial\Omega} \approxeq \frac{1}{2\pi} \frac{\epsilon}{(z-y)^2 + \epsilon^2} \xrightarrow{\epsilon \to 0} \frac{1}{2} \delta_z(y) \ .$$

Hence, $N_\epsilon^{\#} \xrightarrow{\epsilon \to 0} I$.

Have the equation

$$\frac{1}{2}(I + N_\epsilon^{\#})(\partial_\nu w_\epsilon)(z) = \partial_\nu \int_{M_\epsilon} G_0(z; y)(f_1(y) \log \epsilon + f_2(y)) \mathrm{dVol}_{\bar{g}}(y)$$

Completing the proof

Lemma

The right-hand side

$$g_\epsilon(z) := \partial_\nu \int_{M_\epsilon} G_0(z;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y) = \mathcal{O}(\log\epsilon) \quad \textit{as } \epsilon \to 0.$$

Conclusion (Above lemma & layer potential techniques):

$$\partial_\nu w_\epsilon(z) = \left(\frac{1}{2}(I + N_\epsilon^{\#})\right)^{-1} g_\epsilon(z) = \mathcal{O}(\log\epsilon) \quad \text{as } \epsilon \to 0.$$

Proof main theorem

By Green's theorem, $w_\epsilon(x)$ satisfies:

$$\begin{aligned} w_\epsilon(x) &= \int_{M_\epsilon} G_0(x;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y) - \int_{\partial M_\epsilon} G_0(x;y)\partial_\nu w_\epsilon(y)\mathrm{d}S(y) \\ &= (\chi(x)-1)\log\epsilon + \mathcal{O}(1) \quad \text{as } \epsilon \to 0 \end{aligned}$$

Proof of the main theorem

By construction:

$$\begin{aligned} u_\epsilon(x) &= w_\epsilon(x) + \chi(x)\log(\rho(x)/\epsilon) \\ &= (\chi(x)-1)\log\epsilon + \chi(x)(\log\rho(x) - \log\epsilon) + \mathcal{O}(1) \\ &= -\log\epsilon + \mathcal{O}(1) \end{aligned}$$

□

Thanks

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