Mean first escape time on asymptotically hyperbolic surfaces

Emanuel József Godfried
j.w. Jesse Gell-Redman, Justin Tzou, Leo Tzou

School of Mathematics and Statistics
The University of Melbourne

APICDE-II
Macquarie University

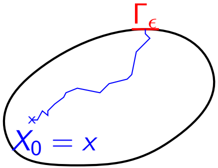
2nd of February 2026

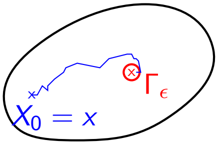
Contents

Contents

Let (M,g) be a Riemannian manifold, we analyse Brownian motion on M

- Mean first escape times (MFET) of BM
- Semi-formally: *Given a Brownian motion X_t starting at $X_0 = x$ and a target Γ_ϵ, what is the mean first escape time of the Brownian motion X_t into Γ_ϵ, and what are the asymptotics as $\epsilon \to 0$?*





Has a rich literature based on probabilistic, PDE, matched asymptotic and microlocal techniques.

Background II

- Given a Riemannian manifold (M, g), Brownian motion is a continuous random variable $X_t, t \geq 0$.
- If X_t starts at $x_0 \in M$ (i.e. $X_0 = x_0$), then the probability density function $f(t,x)$ of X_t is given by the fundamental solution of the heat equation

$$\frac{\partial f(t,x)}{\partial t} = \Delta_g f(t,x) \ , \qquad f(0,x) = \delta_{x_0}(x) \ . \tag{1}$$

- Given a *trap* $\Gamma_\epsilon \subset M/\partial M$, denote the first escape time τ_ϵ of a Brownian motion X_t into Γ_ϵ

$$\tau_\epsilon = \inf\{t \geq 0 : X_t \in \Gamma_\epsilon\} \ .$$

- Interested in the mean first escape time $u_\epsilon(x)$

$$u_\epsilon(x) = \mathbb{E}[\tau_\epsilon : X_0 = x] \tag{2}$$

  and their asymptotics as $\epsilon \to 0$.

- Interested in the mean first escape time $u_\epsilon(x)$

$$u_\epsilon(x) = \mathbb{E}[\tau_\epsilon : X_0 = x] \tag{2}$$

and their asymptotics as $\epsilon \to 0$.

- Integration by parts argument shows

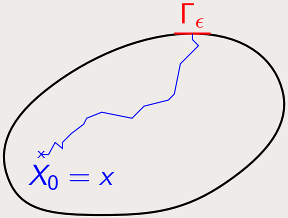
$$\Delta_g u_\epsilon(x) = -1 \quad \text{on } M \setminus \Gamma_\epsilon, \quad u_\epsilon|_{\partial\Gamma_\epsilon} = 0 \tag{3}$$

- Additional conditions can be added e.g. (partially) reflecting boundaries.

Narrow escape in 2D (Holcman & Schuss, 2004)

Let $\Omega \subset \mathbb{R}^2$ bounded with smooth boundary. The mean first escape time $u_\epsilon(x)$ of a Brownian motion (BM) starting at x through a small absorbing trap $\Gamma_\epsilon \subset \partial\Omega$ on an otherwise smooth completely reflecting boundary of radius ϵ satisfies

$$u_\epsilon = -|\Omega| \log \epsilon + \mathcal{O}(1) \quad \text{as } \epsilon \to 0.$$



Introduction & Background IV (Examples and Results)

THE UNIVERSITY OF MELBOURNE

Narrow escape in 3D Riemannian manifolds (Nursultanov, Tzou & Tzou 2021)

$(M^3, \partial M, g)$ a smooth compact Riemannian 3-manifold with boundary. The mean first escape time $u_\epsilon(x)$ of a BM X_t starting at x through a small absorbing trap $\Gamma_\epsilon \subset \partial M$ on an otherwise smooth completely reflecting boundary of radius ϵ satisfies

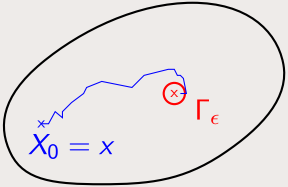
$$u_\epsilon = \frac{|M|}{4\epsilon} - \frac{1}{4\pi} H(x^*)|M| \log \epsilon + \mathcal{O}(1)$$

where $H(x^*)$ is the mean curvature at the centre of the trap.

Narrow capture on general Riemannian surfaces (Nursultanov, Trad, Tzou & Tzou 2023)

Let (M, g) be a smooth compact Riemannian manifold (w, w/o completely reflecting boundary ∂M) of dimension 2. The mean first escape time $u_\epsilon(x)$ of a Brownian motion starting at x into a geodesic ball $\Gamma_\epsilon = B_{x_0}(\epsilon)$ of size ϵ around $x_0 \in M^\circ$ satisfies

$$u_\epsilon(x) = -\frac{|M|_g}{2\pi} \log \epsilon + \mathcal{O}(1) \quad \text{as } \epsilon \to 0.$$



General strategy

Full Cauchy data

For fixed ϵ the solution u_ϵ is uniquely determined by the BVP.
However to find the asymptotics of the solution u_ϵ as $\epsilon \to 0$, the (asymptotics of the) full Cauchy data

$$\mathcal{C}_\epsilon = \{u_\epsilon|_{\partial\Gamma_\epsilon}, = 0, \partial_\nu u_\epsilon|_{\partial\Gamma_\epsilon}\}$$

are needed.
Strategy: Provide asymptotics of the full Cauchy data together with asymptotics of the Green's function to solve the BVP as $\epsilon \to 0$.

Layer potentials

Definition

For $f \in C^\infty(\partial\Omega)$, the single layer potential is the operator

$$\mathrm{SL} : C^\infty(\partial\Omega) \to C^\infty(\Omega) : f \mapsto \int_{\partial\Omega} G(x;y)f(y)\mathrm{d}S(y)$$

Proposition

For $f \in C^\infty(\partial\Omega)$, the single layer potential satisfies the jump relations

$$\lim_{x\to z^+} \mathrm{SL}f(x) = \lim_{x\to z^-} \mathrm{SL}f(x) = \int_{\partial\Omega} G(z;y)f(y)\mathrm{d}S(y)$$

$$\lim_{x\to z^\pm} \partial_\nu \mathrm{SL}f(x) = \frac{1}{2}\left(\mp f(z) + 2\int_{\partial\Omega} \partial_{\nu_z} G(z;y)f(y)\mathrm{d}S(y)\right) = \frac{1}{2}(\mp I + N^\#)f(z)$$

General strategy

Full Cauchy data

For fixed ϵ the solution u_ϵ is uniquely determined by the BVP.
However to find the asymptotics of the solution u_ϵ as $\epsilon \to 0$, the (asymptotics of the) full Cauchy data

$$\mathcal{C}_\epsilon = \{u_\epsilon|_{\Gamma_\epsilon}, = 0, \partial_\nu u_\epsilon|_{\Gamma_\epsilon}\}$$

are needed.
Strategy: Provide asymptotics of the full Cauchy data together with asymptotics of the Green's function to solve the BVP as $\epsilon \to 0$.

General theme of the previous results

Singularity structure of the Green's function $G(x; y)$ are approximately the asymptotics of u_ϵ as $\epsilon \to 0$.

Contents

Background V

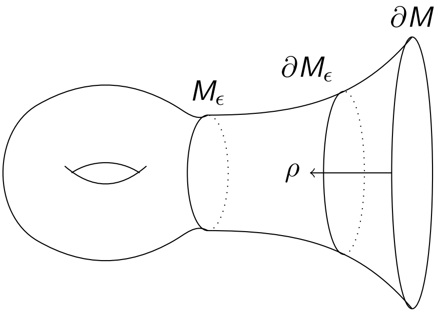
Asymptotically hyperbolic & gas giant manifolds

Asymptotically hyperbolic manifolds

- Riemannian manifolds with boundary $(M, \partial M, g)$ where $g = \bar{g}/\rho^2$ for some boundary defining function ρ, and some compact metric $\bar{g}$.
  - Natural extension the hyperbolic spaces $\mathbb{H}^n$
  - Negatively curved (near the boundary ∂M)
  - Infinite length geodesics and infinite volume

Gas giant manifolds

- Riemannian manifolds with boundary $(M, \partial M, g)$ where $g = \bar{g}/\rho^\alpha$ for some boundary defining function ρ, some compact metric $\bar{g}$ and some parameter $\alpha \in (0, 2)$.
  - Introduced recently
  - Finite length geodesics, but (depending on the dimension and parameter α) infinite volume



- Normalise $|\mathrm{d}\rho|_{\partial M} = 1 > 0$
- Let $\Gamma_\epsilon = \{x \in M : \rho(x) \leq \epsilon\}$, $M_\epsilon := M \setminus \Gamma_\epsilon = \{x \in M : \rho(x) > \epsilon\}$.
- Mean first escape time $u_\epsilon(x)$ of a Brownian motion X_t starting at $x \in M_\epsilon$ into Γ_ϵ satisfies the PDE

$$\Delta_g u_\epsilon(x) = -1 \quad \text{on } M_\epsilon, \quad u_\epsilon|_{\partial M_\epsilon} = 0 \tag{4}$$

Contents

Theorems

MFET of BM on AH and gas giant surfaces

Theorem (J. Gell-Redman, EJG, J. Tzou, L. Tzou)

If $(M, \partial M, g)$ is an asymptotically hyperbolic or gas giant surface, then the mean first escape time from the manifold $M_\epsilon = \{x \in M : \rho(x) \geq \epsilon\}$ satisfies

$$u_\epsilon(x) = \begin{cases} -\log \epsilon + \mathcal{O}(1) & \text{if } M \text{ is asymptotically hyperbolic} \\ \mathcal{O}(1) & \text{if } M \text{ is a gas giant} \end{cases} \quad \text{as } \epsilon \to 0.$$

Strategy I

Lemma

Let M be a two dimensional Riemannian manifold with two metrics g_1, g_2 conformally related by $g_2 = \beta(x) g_1$ for some positive smooth function $\beta(x)$, then

$$\Delta_{g_2} = \Delta_{\beta(x) g_1} = \beta(x) \Delta_{g_1}$$

Corollary

Let $(M, g = \bar{g}/\rho^2)$ be asymptotically hyperbolic, then the boundary value problem (4) is equivalent to

$$\Delta_{\bar{g}} u_\epsilon(x) = -\frac{1}{\rho^2(x)} \quad on\ M_\epsilon, \quad u_\epsilon|_{\partial M_\epsilon} = 0 \ .$$

Strategy II

Let $v_\epsilon(x) \in C^\infty(M_\epsilon)$ be defined by

$$v_\epsilon(x) := \log \rho(x) - \log \epsilon \tag{5}$$

Lemma

If $(M = \mathbb{D}, g = 4\overline{g}_{\mathrm{Euc}}/(1 - x^2 - y^2)^2)$ *is the (constantly negatively curved) Poincaré disc with boundary defining function* $\rho(x) = \frac{1}{2}(1 - x^2 - y^2)$*, then* $v_\epsilon(x)$ *solves the boundary value problem (4), i.e. satisfies the bound* $u_\epsilon(x) = -\log \epsilon + \mathcal{O}(1)$ *as* $\epsilon \to 0$*.*

Strategy

On general (M, g) asymptotically hyperbolic, near the boundary $v_\epsilon(x)$ solves the boundary value problem (4) (approximately). Take a bump-function $\chi(\rho(x)) \equiv 1$ near the boundary ∂M.
If $u_\epsilon(x)$ solves the boundary value problem (4), set

$$w_\epsilon(x) := u_\epsilon(x) - \chi(x)v_\epsilon(x) \tag{6}$$

*and show asymptotics of **error term** $w_\epsilon(x)$.*

We will use Green's formula for this:

$$w_\epsilon(x) = \int_{M_\epsilon} G_0(x; y)\Delta_{\bar{g}} w_\epsilon(y)\mathrm{dVol}_{\bar{g}}(y) - \int_{\partial M_\epsilon} G_0(x; y)\partial_\nu w_\epsilon(y)\mathrm{dS}(y)$$

Proposition

The function w_ϵ satisfies

$$\Delta_{\bar{g}} w_\epsilon(x) = f_1(x)\log\epsilon + f_2(x) \quad \textit{for } x \in M_\epsilon$$

with

- $f_1, f_2 \in C^\infty(M_\epsilon)$
- *If $G_0(x;y)$ is the Dirichlet Green's function for $\Delta_{\bar{g}}$ on $\overline{M}$, then*

$$\int_{M_\epsilon} G_0(x;y) f_1(y) \mathrm{dVol}_{\bar{g}}(y) = \chi(x) - 1$$

- *We have for each $x \in M^\circ$*

$$\int_{M_\epsilon} G_0(x;y) f_2(y) \mathrm{dVol}_{\bar{g}}(y) = \mathcal{O}(1) \quad \textit{as } \epsilon \to 0.$$

Layer potentials

Definition

For $f \in C^\infty(\partial\Omega)$, the single layer potential is the operator

$$\mathrm{SL} : C^\infty(\partial\Omega) \to C^\infty(\Omega) : f \mapsto \int_{\partial\Omega} G(x;y) f(y) \mathrm{d}S(y)$$

Proposition

For $f \in C^\infty(\partial\Omega)$, the single layer potential satisfies the jump relations

$$\lim_{x \to z^+} \mathrm{SL}f(x) = \lim_{x \to z^-} \mathrm{SL}f(x) = \int_{\partial\Omega} G(z;y) f(y) \mathrm{d}S(y)$$

$$\lim_{x \to z^\pm} \partial_\nu \mathrm{SL}f(x) = \frac{1}{2}\left(\mp f(z) + 2\int_{\partial\Omega} \partial_{\nu_z} G(z;y) f(y) \mathrm{d}S(y)\right) = \frac{1}{2}(\mp I + N^\#) f(z)$$

Layer potentials II

By Green's formula we have

$$w_\epsilon(x) = \int_{M_\epsilon} G_0(x;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y) - \int_{\partial M_\epsilon} G_0(x;y)\partial_\nu w_\epsilon(y)\mathrm{d}S(y)$$

Goal: Find asymptotics of $\partial_\nu w_\epsilon(y)$ as $\epsilon \to 0$.
Apply ∂_ν:

$$\partial_\nu w_\epsilon(z) + \partial_\nu \int_{\partial M_\epsilon} G_0(x;y)\partial_\nu w_\epsilon(y)\mathrm{d}S(y) = \partial_\nu \int_{M_\epsilon} G_0(x;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y)$$

By layer potentials, get the equation

$$\frac{1}{2}(I + N_\epsilon^\#)(\partial_\nu w_\epsilon)(z) = \partial_\nu \int_{M_\epsilon} G_0(z;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y)$$

Lemma (Informally)

If $\Omega = \mathbb{D}_{\rho \geq \epsilon} \subset \mathbb{D}$*, and* $G_0(x; y)$ *is the Dirichlet Green's function on* $(\mathbb{D}, g_{\mathrm{Euc}})$*, then*

$$\partial_{\nu_z} G_0(z; y)|_{y \in \partial\Omega} \approxeq \frac{1}{2\pi} \frac{\epsilon}{(z-y)^2 + \epsilon^2} \xrightarrow{\epsilon \to 0} \frac{1}{2} \delta_z(y) \ .$$

Hence, $N_\epsilon^\# \xrightarrow{\epsilon \to 0} I$*.*

Have the equation

$$\frac{1}{2}(I + N_\epsilon^\#)(\partial_\nu w_\epsilon)(z) = \partial_\nu \int_{M_\epsilon} G_0(z; y)(f_1(y) \log \epsilon + f_2(y)) \mathrm{dVol}_{\bar{g}}(y)$$

Lemma

The right-hand side

$$g_\epsilon(z) := \partial_\nu \int_{M_\epsilon} G_0(z;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y) = \mathcal{O}(\log\epsilon) \quad \textit{as } \epsilon \to 0.$$

Conclusion (Above lemma & layer potential techniques):

$$\partial_\nu w_\epsilon(z) = \left(\frac{1}{2}(I + N_\epsilon^{\#})\right)^{-1} g_\epsilon(z) = \mathcal{O}(\log\epsilon) \quad \text{as } \epsilon \to 0.$$

Proof main theorem

By Green's theorem, $w_\epsilon(x)$ satisfies:

$$
\begin{aligned}
w_\epsilon(x) &= \int_{M_\epsilon} G_0(x;y)(f_1(y)\log\epsilon + f_2(y))\mathrm{dVol}_{\bar{g}}(y) - \int_{\partial M_\epsilon} G_0(x;y)\partial_\nu w_\epsilon(y)\mathrm{d}S(y) \\
&= (\chi(x)-1)\log\epsilon + \mathcal{O}(1) + \mathcal{O}(\epsilon\log\epsilon) \quad \text{as } \epsilon\to 0
\end{aligned}
$$

Proof of the main theorem

By construction:

$$
\begin{aligned}
u_\epsilon(x) &= w_\epsilon(x) + \chi(x)\log(\rho(x)/\epsilon) \\
&= (\chi(x)-1)\log\epsilon + \chi(x)(\log\rho(x) - \log\epsilon) + \mathcal{O}(1) \\
&= -\log\epsilon + \mathcal{O}(1)
\end{aligned}
$$

□

Contents

MFET of BM on AH and gas giant surfaces

Theorem (J. Gell-Redman, EJG, J. Tzou, L. Tzou)

If $(M, \partial M, g)$ is an asymptotically hyperbolic or gas giant surface, then the mean first escape time from the manifold $M_\epsilon = \{x \in M : \rho(x) \geq \epsilon\}$ satisfies

$$u_\epsilon(x) = \begin{cases} -\log \epsilon + \mathcal{O}(1) & \text{if } M \text{ is asymptotically hyperbolic} \\ \mathcal{O}(1) & \text{if } M \text{ is a gas giant} \end{cases} \quad \text{as } \epsilon \to 0.$$

The $\mathcal{O}(1)$-number is dependent on the parameter α and not uniform as $\alpha \to 2$.

$$u_{\epsilon,\alpha} \simeq \frac{1}{(2-\alpha)} \left(\rho^{2-\alpha} - \epsilon^{2-\alpha}\right)$$

What about gas giants

MFET on gas giants approaching AH surfaces

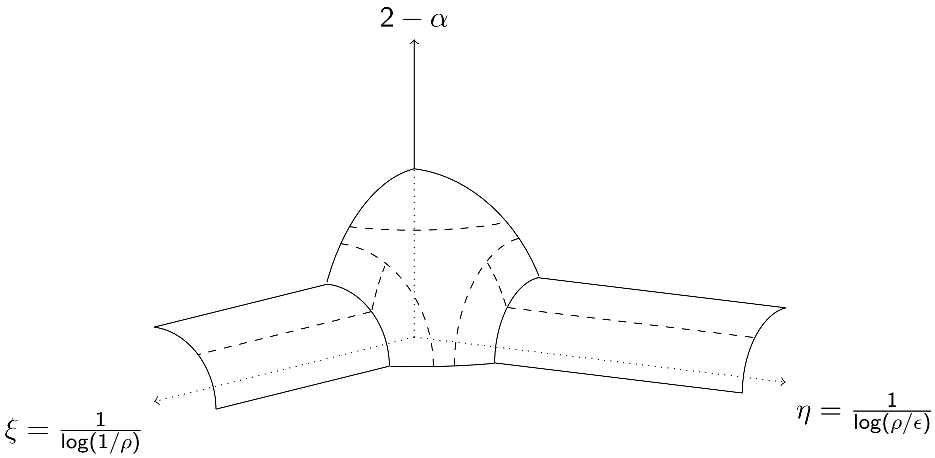
Theorem (J Gell-Redman, EJG, J. Tzou, L. Tzou)

If $(M = \mathbb{D}, g = 2^\alpha \overline{g}_{\mathrm{Euc}}/(1 - x^2 - y^2)^\alpha)$ *is the unit disc with a family of gas giant metrics of parameters* α *approaching 2, then the solutions* $u_{\epsilon,\alpha}(x)$ *approach the solution*

$$u_{\epsilon,2}(x) = -\log \epsilon + \log \rho(x)$$

on the asymptotically hyperbolic surface in a suitable way.

Proof



The function $u_{\epsilon,\alpha}$ is polyhomogeneous conormal on the blow-up space above and approaches $u_{\epsilon,2}$ on the boundary hypersurface $2-\alpha=0$.

Contents

MFET on AH manifolds in higher dimensions

Conjecture (M. Doll, EJG, J. Tzou, L. Tzou)

If $(M, \partial M, g)$ is an asymptotically hyperbolic or gas giant manifold of dimension $n \geq 3$, then the mean first escape time from the manifold $M_\epsilon = \{x \in M : \rho(x) \geq \epsilon\}$ satisfies

$$u_\epsilon(x) = \begin{cases} \mathcal{O}(-\log \epsilon) & \text{if } M \text{ is asymptotically hyperbolic} \\ \mathcal{O}(1) & \text{if } M \text{ is a gas giant} \end{cases} \quad \text{as } \epsilon \to 0.$$

1. Geodesics still reach ∂M_ϵ in $\mathcal{O}(-\log \epsilon)$ time.
2. One "infinite direction". The size of the trap is always infinite for $\epsilon > 0$.
3. On the model manifold $\mathbb{D}^n$ the solution to the PDE is given by $(n-1)\log(\rho/\epsilon)$

Outlook and conjectures

MFET on model gas giants approaching model AH manifolds

Conjecture

If $(M = \mathbb{D}^n, g = 2^\alpha \bar{g}_{\mathrm{Euc}}/(1 - x_1^2 - x_2^2 - \cdots - x_n^2)^\alpha)$ *is the unit ball with a family of gas giant metrics of parameters* α *approaching 2, then the solutions* $u_{\epsilon,\alpha}(x)$ *approach the solution*

$$(n-1)(\log \rho - \log \epsilon)$$

in a suitable way.

MFET on general gas giants approaching general AH manifols

Question: Does the above statement hold for any family of gas giant metrics of parameter α approaching 2? What geometric conditions are required?

Three days on Dynamics, Geometry and Analysis

The University of Melbourne, Australia
May 4-6, 2026

Contact: Xi Geng (xi.geng@unimelb.edu.au),
Leo Tzou (leo.tzou@unimelb.edu.au)

Thanks

School of Mathematics and Statistics
The University of Melbourne

APICDE-II
Macquarie University