Mean first capture times of random particles

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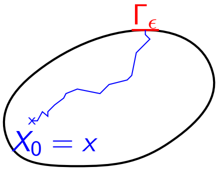
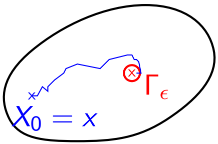
Let (M, g) be a Riemannian manifold, we analyse mean first capture times of random walks on M

- Brownian motion
- Lévy flights

Semi-formally: *Given a random walk X_t starting at $X_0 = x$ and a target Γ_ϵ, what is on average the first time that the random walk X_t is captured into Γ_ϵ, and what are the asymptotics as $\epsilon \to 0$?*





Question: How does the geometry affect the (analysis of the) random walks and the mean first capture time?

Brownian motion

- Given a Riemannian manifold (M, g), Brownian motion is a continuous random variable $X_t, t \geq 0$.
- If X_t starts at $x_0 \in M$ (i.e. $X_0 = x_0$), then the probability density distribution $f(t,x)$ of X_t is given by the fundamental solution of the heat equation

$$\frac{\partial f(t,x)}{\partial t} = \Delta_g f(t,x) \ , \qquad f(0,x) = \delta_{x_0}(x) \ . \tag{1}$$

- Given a *trap* $\Gamma_\epsilon \subset M/\partial M$, denote the first capture time τ_ϵ of a Brownian motion X_t into Γ_ϵ

$$\tau_\epsilon = \inf\left\{t \geq 0 : X_t \in \Gamma_\epsilon\right\} \ .$$

- Interested in the mean first capture time $u_\epsilon(x)$

$$u_\epsilon(x) = \mathbb{E}[\tau_\epsilon : X_0 = x] \tag{2}$$

and their asymptotics as $\epsilon \to 0$.

MFCT of Brownian motion

- Interested in the mean first capture time $u_\epsilon(x)$

$$u_\epsilon(x) = \mathbb{E}[\tau_\epsilon : X_0 = x] \tag{2}$$

and their asymptotics as $\epsilon \to 0$.

- Integration by parts argument shows

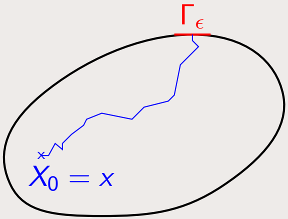
$$\Delta_g u_\epsilon(x) = -1 \quad \text{on } M \setminus \Gamma_\epsilon, \quad u_\epsilon|_{\partial\Gamma_\epsilon} = 0 \tag{3}$$

- Additional conditions can be added e.g. (partially) reflecting boundaries.

Narrow capture in 2D (Holcman & Schuss, 2004)

Let $\Omega \subset \mathbb{R}^2$ bounded with smooth boundary. The mean first capture time $u_\epsilon(x)$ of a Brownian motion (BM) starting at x through a small absorbing trap $\Gamma_\epsilon \subset \partial\Omega$ on an otherwise smooth completely reflecting boundary of radius ϵ satisfies

$$u_\epsilon = -|\Omega| \log \epsilon + \mathcal{O}(1) \quad \text{as } \epsilon \to 0.$$

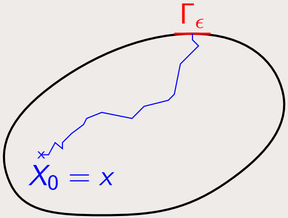


Narrow capture in 3D Riemannian manifolds (Nursultanov, Tzou & Tzou 2021)

$(M^3, \partial M, g)$ a smooth compact Riemannian 3-manifold with boundary. The mean first capture time $u_\epsilon(x)$ of a BM X_t starting at x through a small absorbing trap $\Gamma_\epsilon \subset \partial M$ on an otherwise smooth completely reflecting boundary of radius ϵ satisfies

$$u_\epsilon = \frac{|M|}{4\epsilon} - \frac{1}{4\pi} H(x^*)|M| \log \epsilon + \mathcal{O}(1)$$

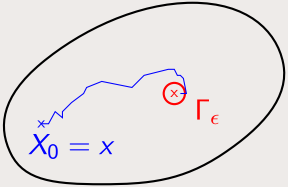
where $H(x^*)$ is the mean curvature at the centre of the trap.



Narrow capture on general Riemannian surfaces (Nursultanov, Trad, Tzou & Tzou 2023)

Let (M, g) be a smooth compact Riemannian manifold (w, w/o completely reflecting boundary ∂M) of dimension 2. The mean first capture time $u_\epsilon(x)$ of a Brownian motion starting at x into a geodesic ball $\Gamma_\epsilon = B_{x_0}(\epsilon)$ of size ϵ around $x_0 \in M^\circ$ satisfies

$$u_\epsilon(x) = -\frac{|M|_g}{2\pi} \log \epsilon + \mathcal{O}(1) \quad \text{as } \epsilon \to 0.$$



General strategy for MFCT of Brownian motion

Full Cauchy data

For fixed ϵ the solution u_ϵ is uniquely determined by the BVP.
However to find the asymptotics of the solution u_ϵ as $\epsilon \to 0$, the (asymptotics of the) full Cauchy data

$$\mathcal{C}_\epsilon = \{u_\epsilon|_{\partial\Gamma_\epsilon}, = 0, \partial_\nu u_\epsilon|_{\partial\Gamma_\epsilon}\}$$

are needed.
Strategy: Provide asymptotics of the full Cauchy data together with asymptotics of the Green's function to solve the BVP as $\epsilon \to 0$.

General theme of the previous results

Singularity structure of the Green's function $G(x; y)$ are approximately the asymptotics of u_ϵ as $\epsilon \to 0$.

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Background V

Asymptotically hyperbolic & gas giant manifolds

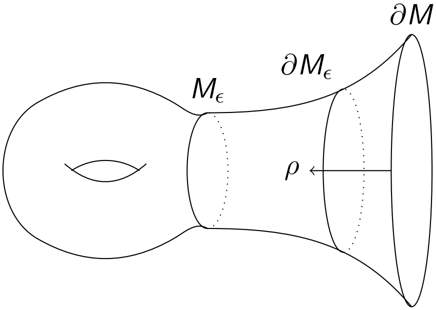
Asymptotically hyperbolic manifolds

- Riemannian manifolds with boundary $(M, \partial M, g)$ where $g = \bar{g}/\rho^2$ for some boundary defining function ρ, and some compact metric $\bar{g}$.
  - Natural extension the hyperbolic spaces $\mathbb{H}^n$
  - Negatively curved (near the boundary ∂M)
  - Infinite length geodesics and infinite volume

Gas giant manifolds

- Riemannian manifolds with boundary $(M, \partial M, g)$ where $g = \bar{g}/\rho^\alpha$ for some boundary defining function ρ, some compact metric $\bar{g}$ and some parameter $\alpha \in (0, 2)$.
  - Introduced recently
  - Finite length geodesics, but (depending on the dimension and parameter α) infinite volume

Setting the setting



- Normalise $|\mathrm{d}\rho|_{\partial M} = 1 > 0$
- Let $\Gamma_\epsilon = \{x \in M : \rho(x) \leq \epsilon\}$, $M_\epsilon := M \setminus \Gamma_\epsilon = \{x \in M : \rho(x) > \epsilon\}$.
- Mean first capture time $u_\epsilon(x)$ of a Brownian motion X_t starting at $x \in M_\epsilon$ into Γ_ϵ satisfies the PDE

$$\Delta_g u_\epsilon(x) = -1 \quad \text{on } M_\epsilon, \quad u_\epsilon|_{\partial M_\epsilon} = 0 \tag{4}$$

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Theorems

Mean first capture time of BM on AH and gas giant surfaces

Theorem (J. Gell-Redman, EJG, J. Tzou, L. Tzou)

If $(M, \partial M, g)$ is an asymptotically hyperbolic or gas giant surface, then the mean first capture time from the manifold $M_\epsilon = \{x \in M : \rho(x) \geq \epsilon\}$ satisfies

$$u_\epsilon(x) = \begin{cases} -\log \epsilon + \mathcal{O}(1) & \text{if } M \text{ is asymptotically hyperbolic} \\ \mathcal{O}(1) & \text{if } M \text{ is a gas giant} \end{cases} \quad \text{as } \epsilon \to 0.$$

Strategy

Let $v_\epsilon(x) \in C^\infty(M_\epsilon)$ be defined by

$$v_\epsilon(x) := \log \rho(x) - \log \epsilon \qquad (5)$$

Lemma

If $(M = \mathbb{D}, g = 4\overline{g}_{\mathrm{Euc}}/(1 - x^2 - y^2)^2)$ is the (constantly negatively curved) Poincaré disc with boundary defining function $\rho(x) = \frac{1}{2}(1 - x^2 - y^2)$, then $v_\epsilon(x)$ solves the boundary value problem (4), i.e. satisfies the bound $u_\epsilon(x) = -\log \epsilon + \mathcal{O}(1)$ as $\epsilon \to 0$.

Strategy II

Strategy

On general (M, g) asymptotically hyperbolic, near the boundary $v_\epsilon(x)$ solves the boundary value problem (4) (approximately). Take a bump-function $\chi(\rho(x)) \equiv 1$ near the boundary ∂M.
If $u_\epsilon(x)$ solves the boundary value problem (4), set

$$w_\epsilon(x) := u_\epsilon(x) - \chi(x) v_\epsilon(x) \tag{6}$$

*and show asymptotics of **error term** $w_\epsilon(x)$.*

The Cauchy data of $w_\epsilon(x)$

Proposition

The full Cauchy data of $w_\epsilon(x)$ satisfies

$$\mathcal{C}_\epsilon(w_\epsilon) = \{w_\epsilon|_{\partial M_\epsilon} = 0, \partial_\nu w_\epsilon|_{\partial M_\epsilon} = \mathcal{O}(-\log \epsilon)\}$$

Proposition

Let $G_0(x;y)$ be the Dirichlet Green's function for $\Delta_{\bar{g}}$, then

$$\int_{M_\epsilon} G_0(x;y)\Delta_{\bar{g}} w_\epsilon(y) \mathrm{dVol}_{\bar{g}}(y) = (\chi - 1)\log \epsilon + \mathcal{O}(1)$$

pointwise.

Proof main theorem

By Green's theorem, $w_\epsilon(x)$ satisfies:

$$\begin{aligned} w_\epsilon(x) &= \int_{M_\epsilon} G_0(x;y)\Delta_{\bar{g}} w_\epsilon(y)\mathrm{dVol}_{\bar{g}}(y) - \int_{\partial M_\epsilon} G_0(x;y)\partial_\nu w_\epsilon(y)\mathrm{d}S(y) \\ &= (\chi(x)-1)\log\epsilon + \mathcal{O}(1) + \mathcal{O}(\epsilon\log\epsilon) \quad \text{as } \epsilon \to 0 \end{aligned}$$

pointwise.

Proof of the main theorem

By construction:

$$\begin{aligned} u_\epsilon(x) &= w_\epsilon(x) + \chi(x)v_\epsilon(x) \\ &= w_\epsilon(x) + \chi(x)[\log(\rho(x)) - \log(\epsilon)] \\ &= (\chi(x)-1)\log\epsilon + \mathcal{O}(1) + \chi(x)[\log\rho(x) - \log\epsilon] \\ &= -\log\epsilon + \mathcal{O}(1) \end{aligned}$$

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What about the gas giants?

Mean first capture time of BM on AH and gas giant surfaces

Theorem (J. Gell-Redman, EJG, J. Tzou, L. Tzou)

If $(M, \partial M, g)$ is an asymptotically hyperbolic or gas giant surface, then the mean first capture time from the manifold $M_\epsilon = \{x \in M : \rho(x) \geq \epsilon\}$ satisfies

$$u_\epsilon(x) = \begin{cases} -\log \epsilon + \mathcal{O}(1) & \text{if } M \text{ is asymptotically hyperbolic} \\ \mathcal{O}(1) & \text{if } M \text{ is a gas giant} \end{cases} \quad \text{as } \epsilon \to 0.$$

The $\mathcal{O}(1)$-number is dependent on the parameter α and not uniform as $\alpha \to 2$.

$$u_{\epsilon,\alpha} \simeq \frac{1}{(2-\alpha)} \left(\rho^{2-\alpha} - \epsilon^{2-\alpha}\right)$$

What about gas giants

MFCT on gas giants approaching AH surfaces

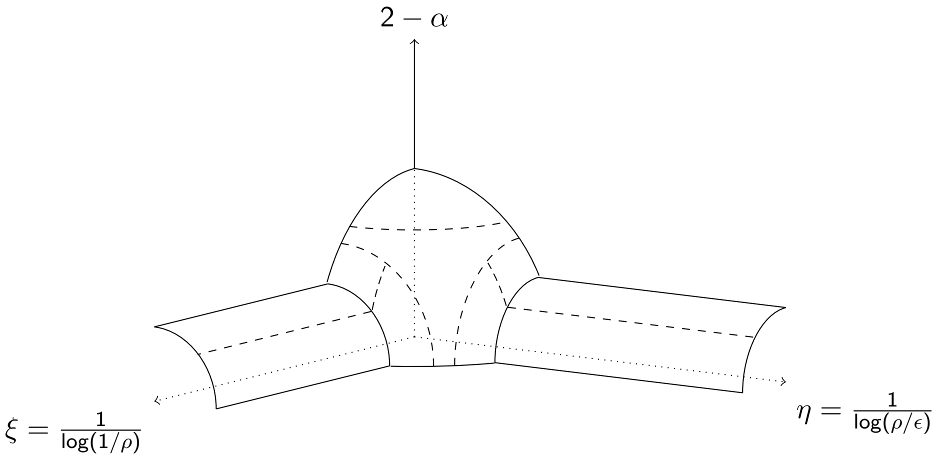
Theorem (J Gell-Redman, EJG, J. Tzou, L. Tzou)

If $(M = \mathbb{D}, g = 2^\alpha \overline{g}_{\mathrm{Euc}}/(1 - x^2 - y^2)^\alpha)$ *is the unit disc with a family of gas giant metrics of parameters* α *approaching 2, then the solutions* $u_{\epsilon,\alpha}(x)$ *approach the solution*

$$u_{\epsilon,2}(x) = -\log \epsilon + \log \rho(x)$$

on the asymptotically hyperbolic surface in a suitable way.

Proof



The function $u_{\epsilon,\alpha}$ is polyhomogeneous conormal on the blow-up space above and approaches $u_{\epsilon,2}$ on the boundary hypersurface $2-\alpha=0$.

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Outlook and conjectures

MFCT on AH manifolds in higher dimensions

Conjecture (M. Doll, EJG, J. Tzou, L. Tzou)

If $(M, \partial M, g)$ is an asymptotically hyperbolic or gas giant manifold of dimension $n \geq 3$, then the mean first capture time from the manifold $M_\epsilon = \{x \in M : \rho(x) \geq \epsilon\}$ satisfies

$$u_\epsilon(x) = \begin{cases} \mathcal{O}(-\log \epsilon) & \text{if } M \text{ is asymptotically hyperbolic} \\ \mathcal{O}(1) & \text{if } M \text{ is a gas giant} \end{cases} \quad \text{as } \epsilon \to 0.$$

1. Geodesics still reach ∂M_ϵ in $\mathcal{O}(-\log \epsilon)$ time.
2. One "infinite direction". The size of the trap is always infinite for $\epsilon > 0$.
3. On the model manifold $\mathbb{D}^n$ the solution to the PDE is given by $(n-1)\log(\rho/\epsilon)$

Outlook and conjectures

MFCT on model gas giants approaching model AH manifolds

Conjecture

If $(M = \mathbb{D}^n, g = 2^\alpha \bar{g}_{\mathrm{Euc}}/(1 - x_1^2 - x_2^2 - \cdots - x_n^2)^\alpha)$ *is the unit ball with a family of gas giant metrics of parameters* α *approaching 2, then the solutions* $u_{\epsilon,\alpha}(x)$ *approach the solution*

$$(n-1)\left(\log \rho - \log \epsilon\right)$$

in a suitable way.

MFCT on general gas giants approaching general AH manifols

Question: Does the above statement hold for any family of gas giant metrics of parameter α approaching 2? What geometric conditions are required?

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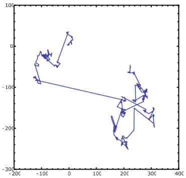
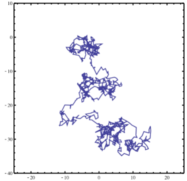
Lévy flights

Wikipedia's definition

A Lévy flight is a random walk in which the step-lengths have a stable distribution, a probability distribution that is heavy-tailed.
When defined as a walk in a space of dimension greater than one, the steps made are in isotropic random directions.



(a) Lévy flights: By PAR - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=9569860

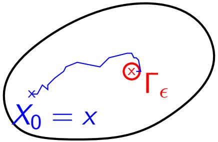


(b) Brownian motion: By PAR - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=9569957

Lévy flight foraging hypothesis

Viswanathan - Raposo - da Luz (2008)

"Since Lévy flights and walks can optimize search efficiencies, therefore natural selection should have led to adaptations for Lévy flight foraging"



- Longer jumps help with searching "different areas"
- Controversial
  - Counter examples have been found where Brownian motion is "better"

Lévy flights on $\mathbb{R}^n$

Brownian motion

The probability density distribution $f(t,x)$ of a Brownian motion starting at x_0 is given by the fundamental solution of the heat equation

$$\frac{\partial f(t,x)}{\partial t} = \Delta_g f(t,x) \; , \qquad f(0,x) = \delta_{x_0}(x) \; .$$

On $\mathbb{R}^n$: the probability density distribution $g(t,x)$ of a Lévy flight starting at x_0 is given by the fundamental solution of the *fractional* heat equation

$$\frac{\partial g(t,x)}{\partial t} = (-\Delta_{\mathbb{R}^n})^{\alpha} g(t,x), \qquad g(0,x) = \delta_{x_0}(x) \tag{7}$$

where for $\alpha \in (0,1)$, the fractional Laplacian can be given by:

$$(-\Delta_{\mathbb{R}^n})^{\alpha} u(x) := \mathcal{F}^{-1}\left[|\xi|^{2\alpha} \hat{u}(\xi) \right] \; ,$$

but can be invariantly defined as a pseudodifferential operator of order 2α.

Lévy flights on manifolds

Obviously: if (M, g) is Riemannian, then the probability density distribution of a Lévy flight starting at $x_0 \in M$ is given by the fundamental solution of the fractional heat equation

$$\frac{\partial g(t,x)}{\partial t} = -(-\Delta_g)^\alpha g(t,x), \qquad g(0,x) = \delta_{x_0}(x)$$

for $\alpha \in (0,1)$.

Wrong! The Lévy flight generator is given by

$$\mathcal{A}_\alpha u(x) = c_{n,\alpha} \mathrm{p.v.} \int_{T_x M \setminus 0} \frac{u(\exp_x(v)) - u(x)}{|v|_g^{n+2\alpha}} \mathrm{d}\, T_x(v)\ . \tag{8}$$

The probability density distribution of a Lévy flight starting at $x_0 \in M$ is given by the fundamental solution of the equation

$$\frac{\partial g(t,x)}{\partial t} = \mathcal{A}_\alpha g(t,x), \qquad g(0,x) = \delta_{x_0}(x) \tag{9}$$

for $\alpha \in (0,1)$.

Key facts

- If $(M,g) = (\mathbb{R}^n, g_{\mathrm{Euc}}), (\mathbb{T}^n, g_{\mathrm{Euc}})$, then $\mathcal{A}_\alpha = -(-\Delta_{\mathbb{R}^n})^\alpha$
- The operators $\mathcal{A}_\alpha, (-\Delta_{\mathbb{R}^n})^\alpha$ are non-local
- If $(M,g) = (S^n, g_{\mathrm{std}})$, then

$$\mathcal{A}_\alpha u(x) = \left[-(-\Delta_{S^n})^\alpha + \mathcal{A}_0 + \mathcal{A}_{-1}\mathcal{J}\right] u(x) \ ,$$

  with $\mathcal{A}_0, \mathcal{A}_{-1} \in \Psi(S^n)$ and $\mathcal{J}u(x) = u(-x)$.
- Singularities of $u(x)$ get propagated to $-x$ under $\mathcal{A}_\alpha$

Mean first capture on spheres

Divergence at the south pole Chaubet-Guedes Bonthonneau-Lefeuvre-Tzou (2025)

If Γ_ϵ is a trap of size ϵ at the north pole N, then the mean first capture time $u_\epsilon(x)$ of a Lévy flight process on S^n into the trap Γ_ϵ satisfies

$$u_\epsilon(S) - |S^n|^{-1} \int_{S^n \setminus \Gamma_\epsilon} u_\epsilon(x) \mathrm{dVol}_{S^n}(x) = -\frac{C_{n,\alpha}|S^n|}{\epsilon^{n-1-4\alpha}} + o(\epsilon^{4\alpha+1-n}) \to -\infty$$

if $n > 1 + 4\alpha$.

Convergence to the spacial mean of BMNursultanov-Trad-Tzou-Tzou (2023)

Let (M, g) be a closed 2-dimensional Riemannian manifold. Let Γ_ϵ be a trap, then the difference from the spacial mean of the mean first capture time of BM into Γ_ϵ satisfies

$$u_\epsilon(x) - |M \setminus \Gamma_\epsilon|_g^{-1} \int_{M \setminus \Gamma_\epsilon} u_\epsilon(y) \mathrm{dVol}_g(y) = \mathcal{O}(\epsilon \log \epsilon)$$

Conjugate points

The exponential map on the sphere

Let (M, g) a Riemannian manifold, the exponential map

$$\exp_x : T_x M \to M$$

is not always (everywhere) a **local** diffeomorphism.

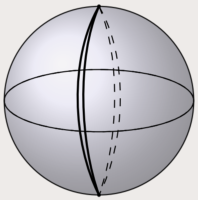


Figure: The north and south poles are conjugate points

Zoll sphere

Definition
A Zoll sphere is a Riemannian manifold (M, g) homeomorphic to S^d such that all the geodesics are closed.

Corollary
All the geodesics have the same length (normalise to 2π).

Corollary
In every direction there is a conjugate vector $v \in T_x M$ (and a conjugate point $y = \exp_x(v)$).

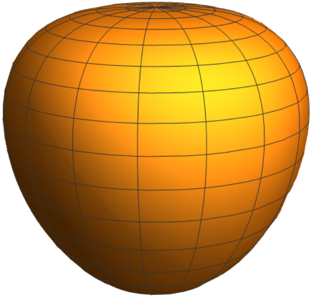


Figure: A Zoll sphere of revolution, embedded in $\mathbb{R}^3$.

Proposition

If M is a Zoll surface, then points q on the conjugate locus of p are of one of the following type

(a) *locally smooth*
(b) *one of a finite number of simple cusps*
(c) *the entire conjugate locus of p (antipodal pts)*

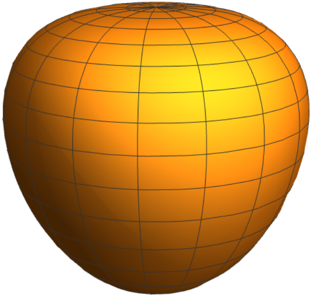
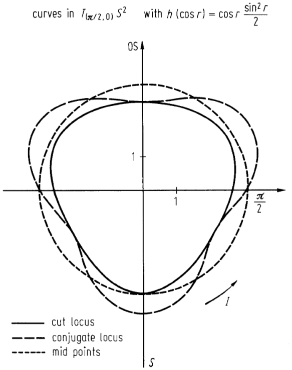


Figure: A Zoll sphere of revolution, embedded in $\mathbb{R}^3$.

Conjugate loci on Zoll surfaces



(a) Conjugate vectors of a point on a Zoll sphere of revolution [Besse, Manifolds all of whose Geodesics are Closed, 4.37].

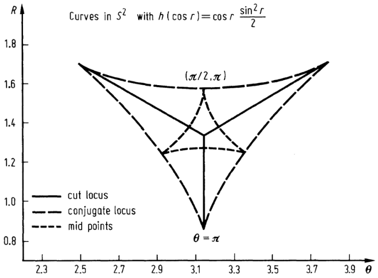


Fig. 4.38

(b) Conjugate locus of a point on a Zoll sphere of revolution [Besse, Manifolds all of whose Geodesics are Closed, 4.38].

Figure: Conjugate vectors and locus of point on the equator of the surface above.

Lévy flights on Zoll surfaces

Proposition

The Lévy flight generator A_α is a sum

$$\mathcal{A}_\alpha = -(-\Delta_g)^\alpha + \mathcal{A}_0 + \mathcal{I}_{-1}$$

with $(-\Delta_g)^\alpha$ the fractional Laplacian, $\mathcal{A}_0 \in \Psi^0(M)$ and $\mathcal{I}_{-1} \in I^{-1}(M, M; \mathcal{C})$

Conjecture (Y. Chaubet, EJG, L. Tzou)

Let p be a point on a Zoll surface, and $q \in \mathrm{Conj}(p)$, the difference from the mean of the MFCT into the target Γ_ϵ around p

$$u_\epsilon(q) - \int_{M \setminus \Gamma_\epsilon} u_\epsilon(x) \mathrm{dVol}_g(x)$$

diverges as $\epsilon \to 0$, and the rate depends on the type of point q is on the conjugate locus (locally smooth, simple cusp, entire conjugate locus).

Thanks

School of Mathematics and Statistics

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