An introduction to narrow capture problems on Riemannian manifolds

South Pacific mathematics exchange seminars

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The University of Melbourne

9 April 2026

The two main characters

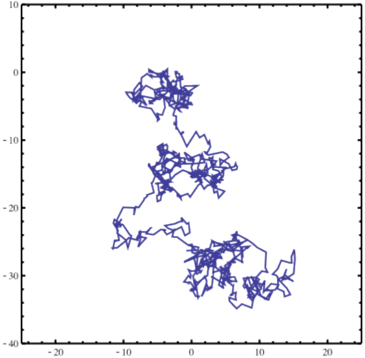

Figure 1: Brownian motion: By PAR - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=9569957

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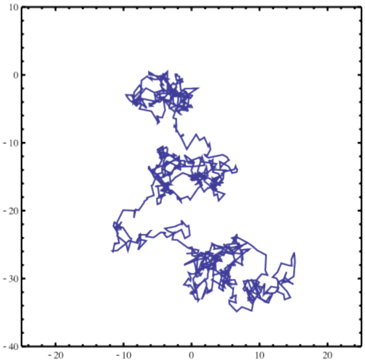

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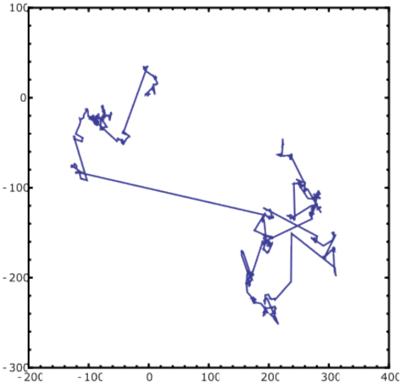
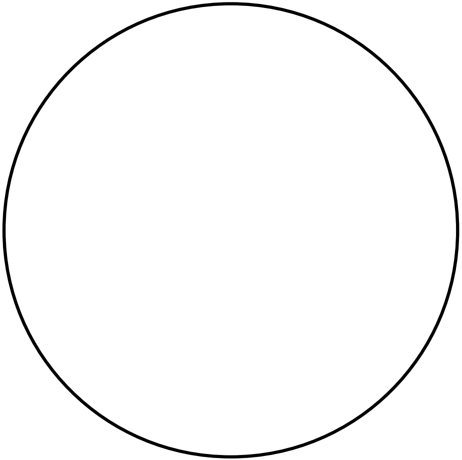


Figure 2: Lévy flights: By PAR - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=9569860

The narrow capture problem on a disc



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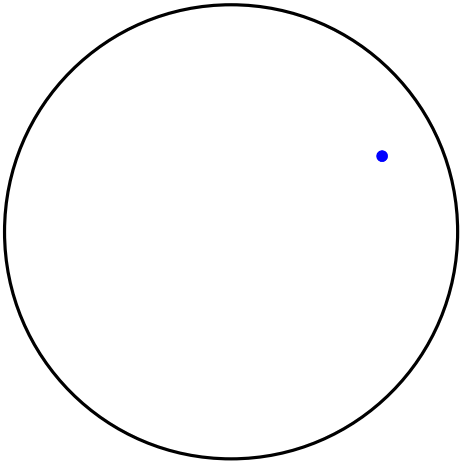

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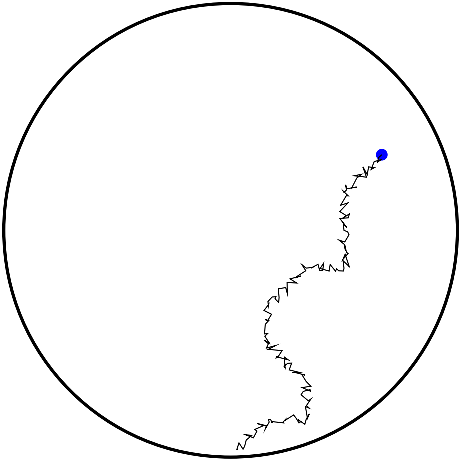

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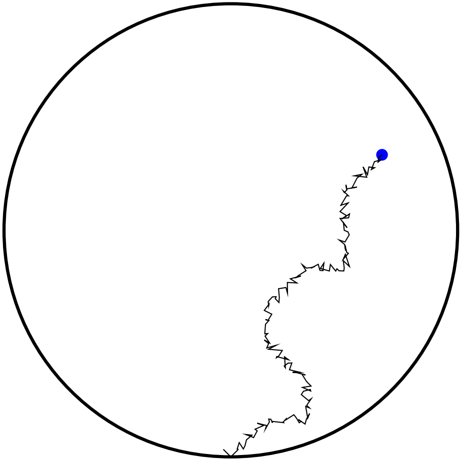

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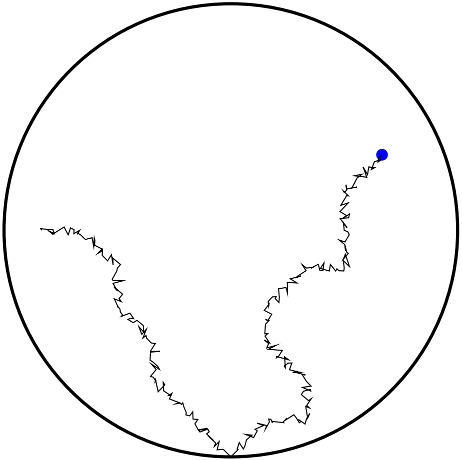

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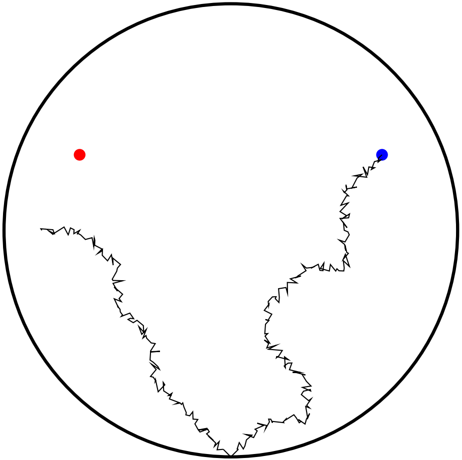

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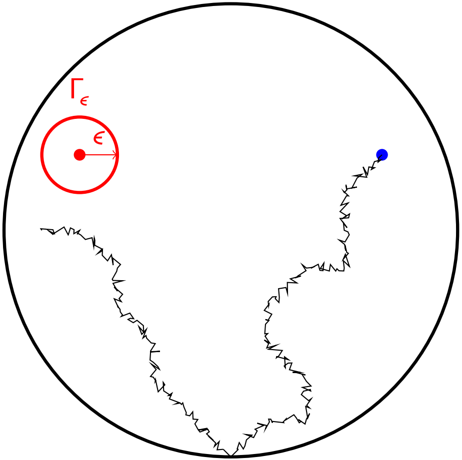

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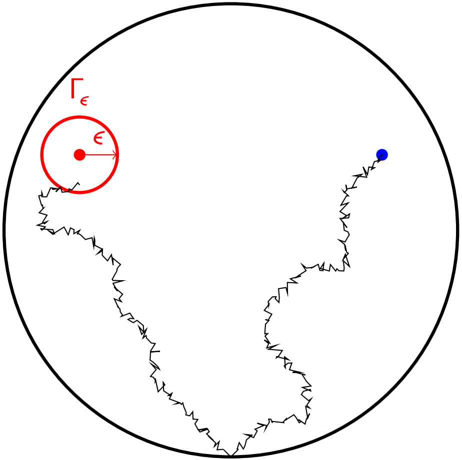


The Lévy flight foraging hypothesis

nature > letters > article

Letter | Published: 30 May 1996

Lévy flight search patterns of wandering albatrosses

G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince & H. E. Stanley

Nature **381**, 413–415 (1996) | Cite this article

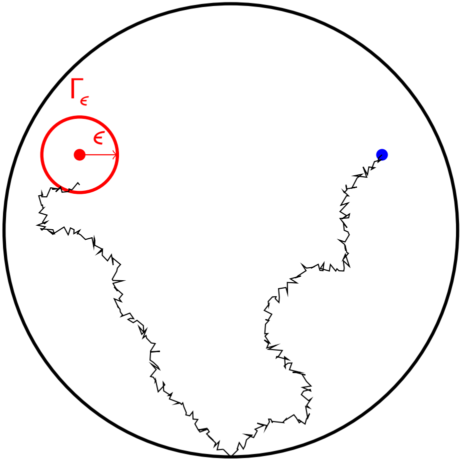
Abstract

LéVY flights are a special class of random walks whose step lengths are not constant but rather are chosen from a probability distribution with a power-law tail. Realizations of Lévy flights in physical phenomena are very diverse, examples including fluid dynamics, dynamical systems, and micelles[1,2]. This diversity raises the possibility that Lévy flights may be found in biological systems. A decade ago, it was proposed that Lévy flights may be observed in the behaviour of foraging ants[3]. Recently, it was argued that *Drosophila* might perform Lévy flights[4], but the hypothesis that foraging animals in natural environments perform Lévy flights has not been tested. Here we study the foraging behaviour of the wandering albatross *Diomedea exulans*, and find a power-law distribution of flight-time intervals. We interpret our finding of temporal scale invariance in terms of a scale-invariant spatial distribution of food on the ocean surface. Finally, we examine the significance of our finding in relation to the basis of scale-invariant phenomena observed in biological systems.

Viswanathan et al. ('08)

Since Lévy flights and walks can optimize search efficiencies, therefore natural selection should have led to adaptations for Lévy flight foraging.

The narrow capture problem on a disc - asymptotics



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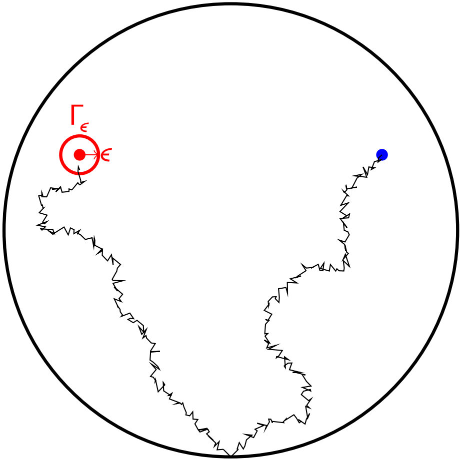

Let X_t a random walk starting at x. Denote:

$$u_\epsilon(x) := \mathbb{E}[t : X_t \text{ in } \Gamma_\epsilon \text{ for the first time}]$$

Cheviakov-Ward-Straube ('10),
Chevalier-Bénichou-Meyer-Voituriez ('10)

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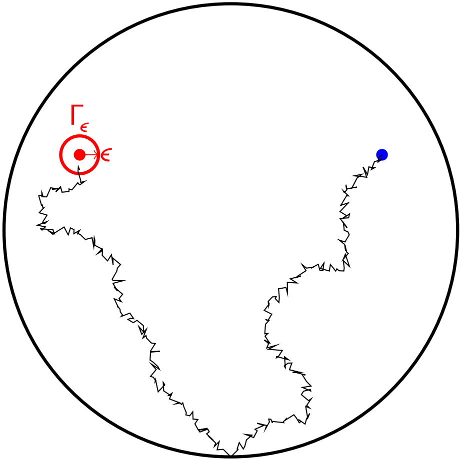

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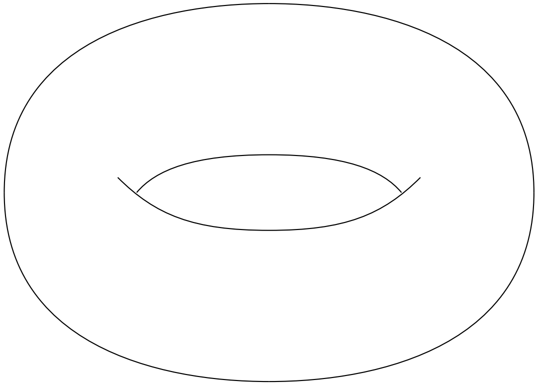
The mean first capture time $u_\epsilon(x)$ of a Brownian particle confined to a disc into a small interior target of size ϵ is given by

$$u_\epsilon(x) \sim \begin{cases} C_2 \cdot (-\log \epsilon) & \text{n} = 2 \\ C_n \epsilon^{2-n} & \text{n} > 2 \end{cases} \quad \text{as } \epsilon \to 0$$

pointwise.

Narrow capture problem on a Riemannian manifold



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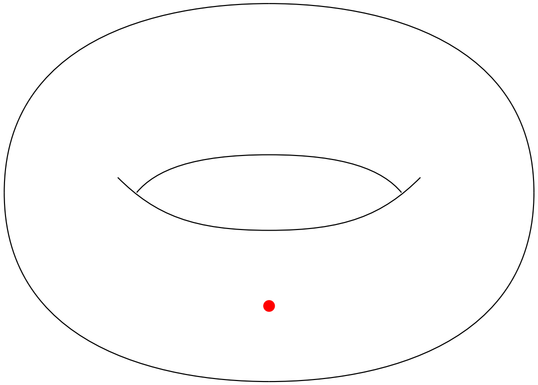

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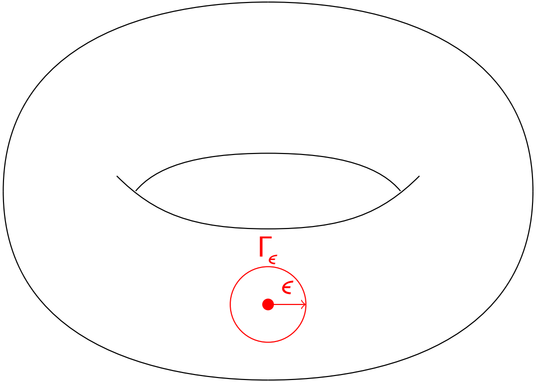

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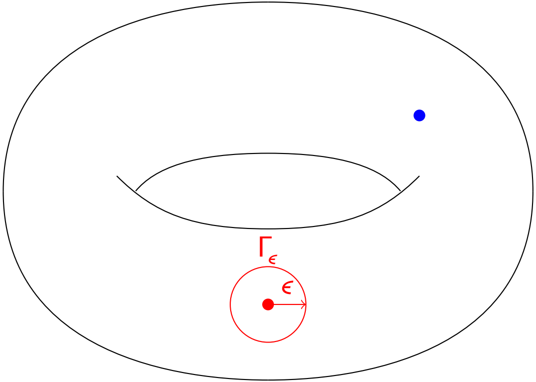

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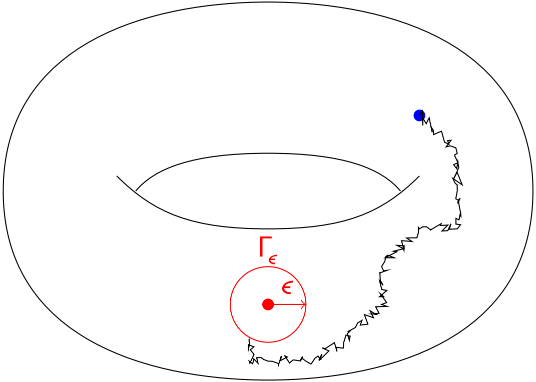

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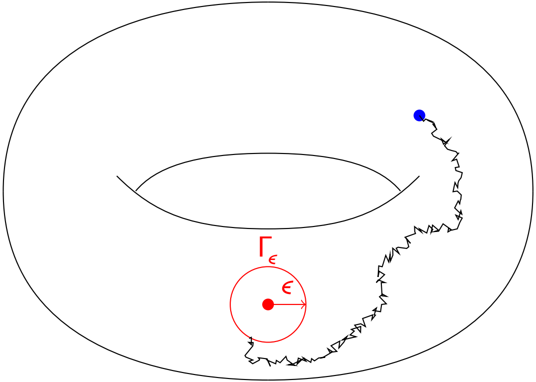

Nursultanov-Trad-Tzou-Tzou ('23)

The mean first capture time of a Brownian particle into a trap of size ϵ satisfies

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The Lévy flight foraging hypothesis - a controversy

Letter | Published: 28 February 2008

Scaling laws of marine predator search behaviour

David W. Sims, Emily J. Southall, Nicolas E. Humphries, Graeme C. Hays, Corey J. A. Bradshaw, Jonathan W. Pitchford, Alex James, Mohammed Z. Ahmed, Andrew S. Brierley, Mark A. Hindell, David Morritt, Michael K. Musyl, David Righton, Emily L. C. Shepard, Victoria J. Wearmouth, Rory P. Wilson, Matthew J. Witt & Julian D. Metcalfe

Nature **451**, 1098–1102 (2008) | Cite this article

Abstract

Many free-ranging predators have to make foraging decisions with little, if any, knowledge of present resource distribution and availability[1]. The optimal search strategy they should use to maximize encounter rates with prey in heterogeneous natural environments remains a largely unresolved issue in ecology[1,2,3]. Lévy walks[4] are specialized random walks giving rise to fractal movement trajectories that may represent an optimal solution for searching complex landscapes[5]. However, the adaptive significance of this putative strategy in response to natural prey distributions remains untested[6,7]. Here we analyse over a million movement displacements recorded from animal-attached electronic tags to show that diverse marine predators—sharks, bony fishes, sea turtles and penguins—exhibit Lévy-walk-like behaviour close to a theoretical optimum[2]. Prey density distributions also display Lévy-like fractal patterns, suggesting response movements by predators to prey distributions. Simulations show that predators have higher encounter rates when adopting Lévy-type foraging in natural-like prey fields compared with purely random landscapes. This is consistent with the hypothesis that observed search patterns are adapted to observed statistical patterns of the landscape. This may explain why Lévy-like behaviour seems to be widespread among diverse organisms[3], from microbes[8] to humans[9], as a 'rule' that evolved in response to patchy resource distributions.

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Letter | Published: 25 October 2007

Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer

Andrew M. Edwards, Richard A. Phillips, Nicholas W. Watkins, Mervyn P. Freeman, Eugene J. Murphy, Vsevolod Afanasyev, Sergey V. Buldyrev, M. G. E. da Luz, E. P. Raposo, H. Eugene Stanley & Gandhimohan M. Viswanathan

Nature **449**, 1044–1048 (2007) | Cite this article

Abstract

The study of animal foraging behaviour is of practical ecological importance[1], and exemplifies the wider scientific problem of optimizing search strategies[2]. Lévy flights are random walks, the step lengths of which come from probability distributions with heavy power-law tails[3,4], such that clusters of short steps are connected by rare long steps. Lévy flights display fractal properties, have no typical scale, and occur in physical[3,4,5] and chemical[6] systems. An attempt to demonstrate their existence in a natural biological system presented evidence that wandering albatrosses perform Lévy flights when searching for prey on the ocean surface[7]. This well known finding[2,4,8,9] was followed by similar inferences about the search strategies of deer[10] and bumblebees[10]. These pioneering studies have triggered much theoretical work in physics (for example, refs 11, 12), as well as empirical ecological analyses regarding reindeer[13], microzooplankton[14], grey seals[15], spider monkeys[16] and fishing boats[17]. Here we analyse a new, high-resolution data set of wandering albatross flights, and find no evidence for Lévy flight behaviour. Instead we find that flight times are gamma distributed, with an exponential decay for the longest flights. We re-analyse the original albatross data[7] using additional information, and conclude that the extremely long flights,

The Lévy flight foraging hypothesis - a controversy

What does it mean to *optimize search efficiencies*?

Narrow capture of Brownian motion

Narrow capture of Lévy flights on S^n, T^n

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Asymptotically Brownian motion can be better!

Two types of random walks

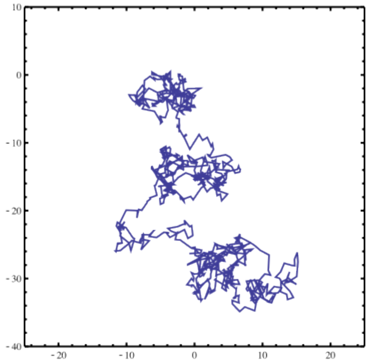


Figure 3: Brownian motion: By PAR - Own work, Public Domain, `https://commons.wikimedia.org/w/index.php?curid=9569957`

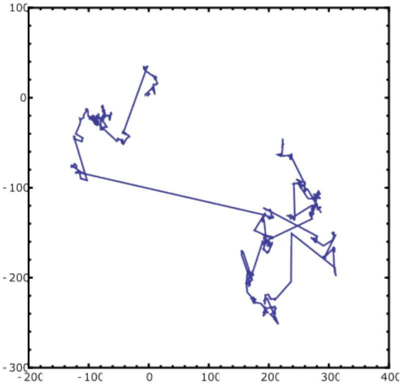
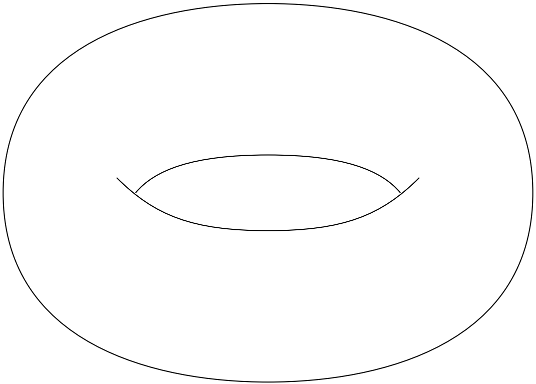


Figure 4: Lévy flights: By PAR - Own work, Public Domain, `https://commons.wikimedia.org/w/index.php?curid=9569860`

Generating random walks - Brownian motion



Generating a Brownian motion $\{X_t\}_{t\geq 0}$

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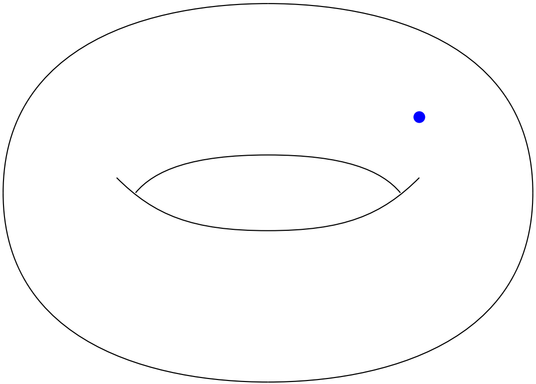
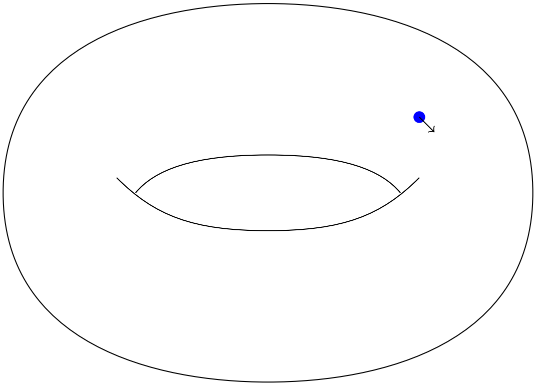


Generating a Brownian motion $\{X_t\}_{t \geq 0}$

- Let our particle be at position x_t

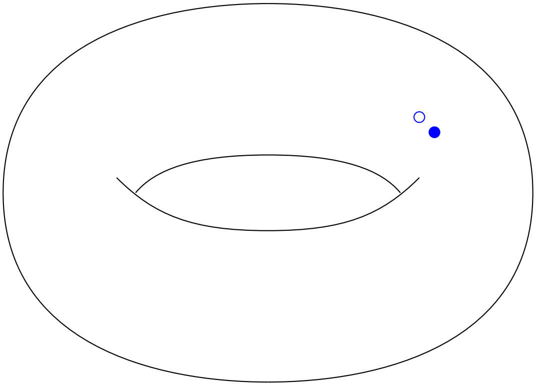
Generating random walks - Brownian motion



Generating a Brownian motion $\{X_t\}_{t\geq 0}$

- Let our particle be at position x_t
- Randomly choose a unit direction vector v

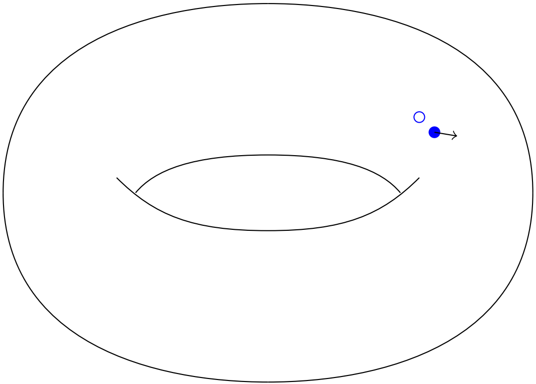
Generating random walks - Brownian motion



Generating a Brownian motion $\{X_t\}_{t\geq 0}$

- Let our particle be at position x_t
- Randomly choose a unit direction vector v
- Take a step of size h in that direction

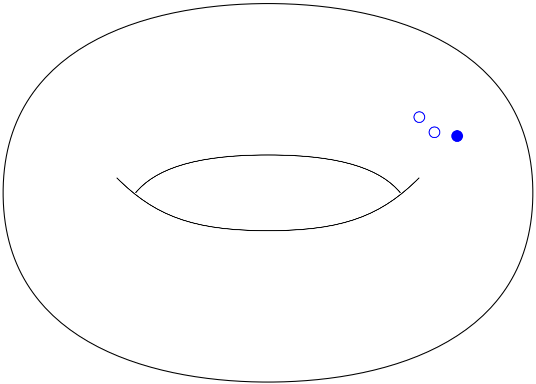
Generating random walks - Brownian motion



Generating a Brownian motion $\{X_t\}_{t\geq 0}$

- Let our particle be at position x_t
- Randomly choose a unit direction vector v
- Take a step of size h in that direction
- Be at position x_{t+h^2}

Generating random walks - Brownian motion



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- Let our particle be at position x_t
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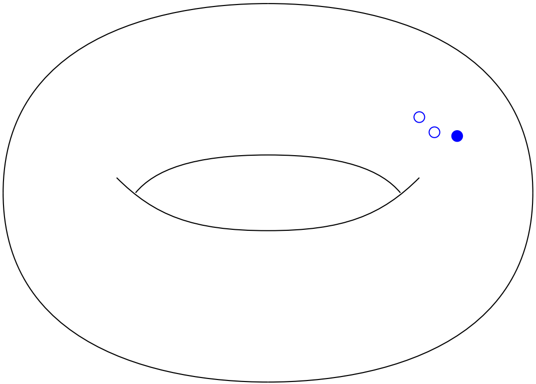
- Repeat

Generating random walks - Brownian motion



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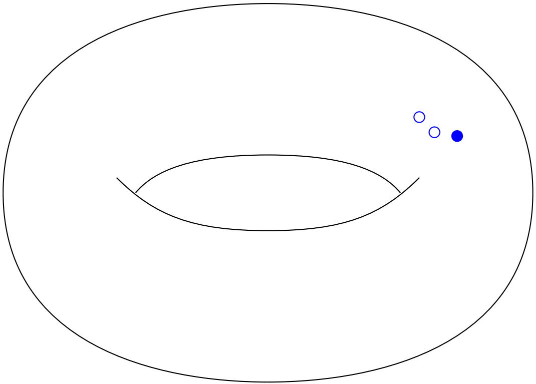

Generating a Brownian motion $\{X_t\}_{t\geq 0}$

- Let our particle be at position x_t
- Randomly choose a unit direction vector v
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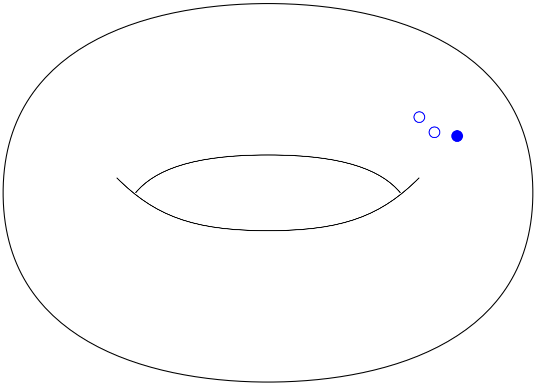
- Repeat

Take step size $h \to 0$, to get a continuous random walk $\{X_t\}_{t\geq 0}$.

Generating random walks - Brownian motion



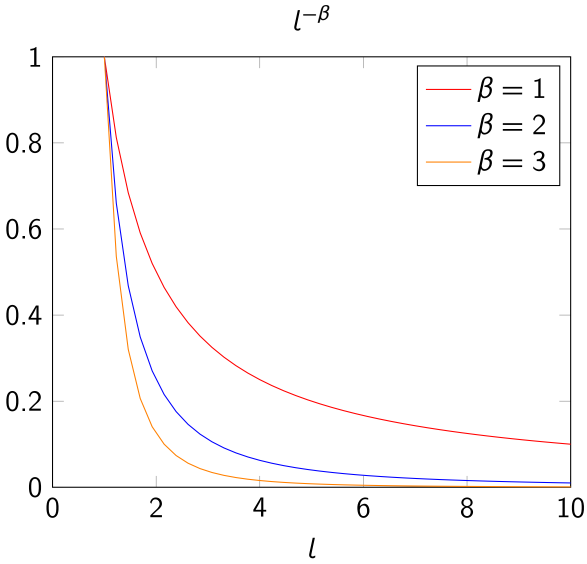
Generating a Brownian motion $\{X_t\}_{t\geq 0}$

- Let our particle be at position x_t
- Randomly choose a unit direction vector v
- Take a step of size h in that direction
- Be at position x_{t+h^2}
- Repeat

Take step size $h \to 0$, to get a continuous random walk $\{X_t\}_{t\geq 0}$.
For any step: only LOCAL geometry

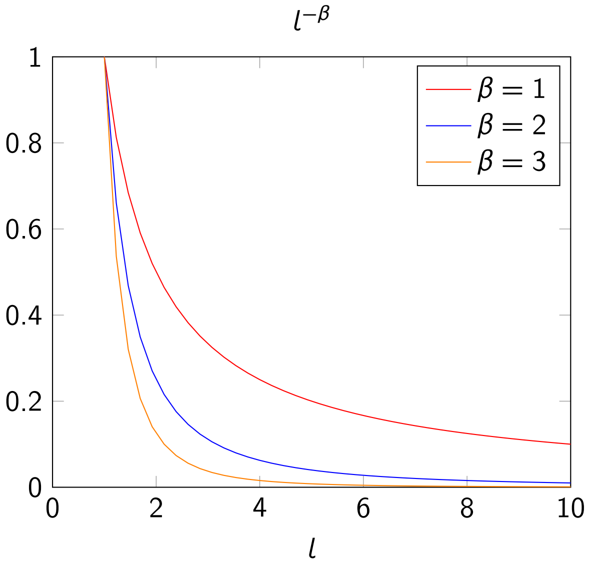
Generating random walks - Lévy flights

- Choose step lengths according to a power-law distribution



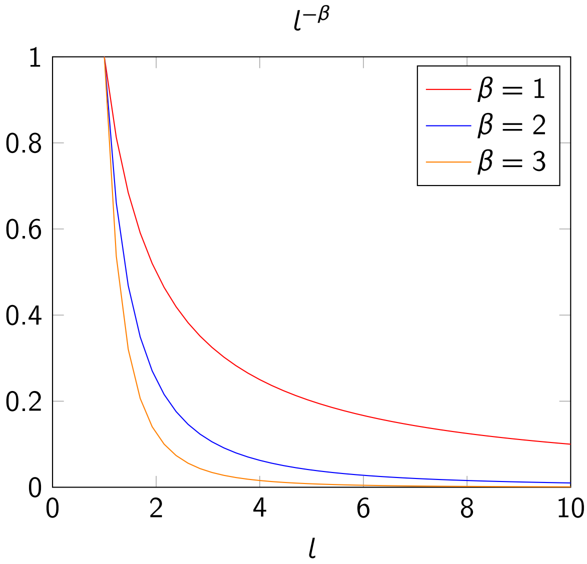
Generating random walks - Lévy flights

- Choose step lengths according to a power-law distribution
- $l^{-1-2\alpha} dl$ for $\alpha \in (0, 1)$



Generating random walks - Lévy flights

- Choose step lengths according to a power-law distribution
- $l^{-1-2\alpha}dl$ for $\alpha \in (0,1)$
- Long jumps



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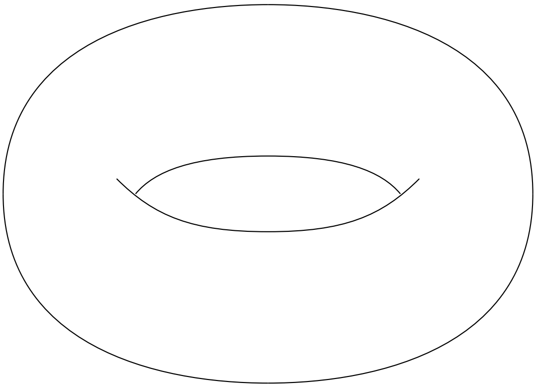

Generating a Lévy flight $\{Y_t\}_{t\geq 0}$ of parameter $\alpha \in (0,1)$

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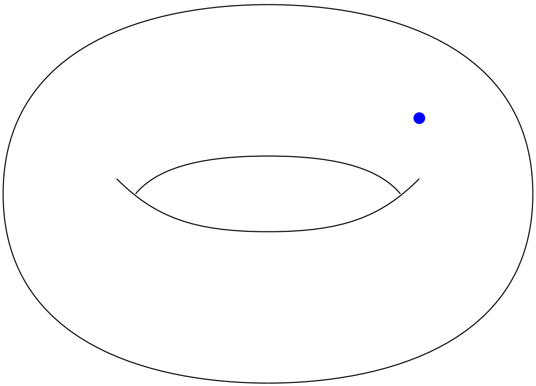

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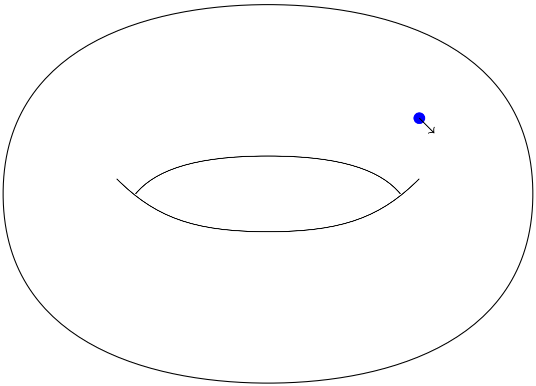
Generating random walks - Lévy flights



Generating a Lévy flight $\{Y_t\}_{t\geq 0}$ of parameter $\alpha \in (0,1)$

- Let our particle be at position y_t
- Randomly choose a unit direction vector v

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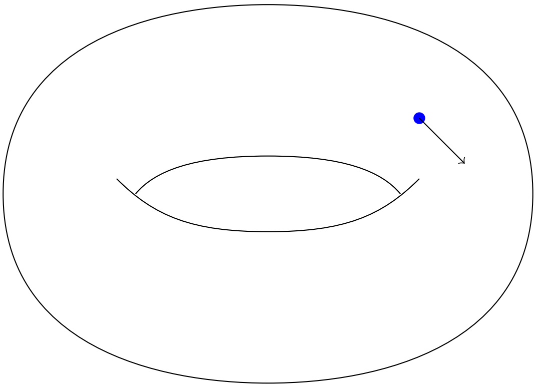

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- Let our particle be at position y_t
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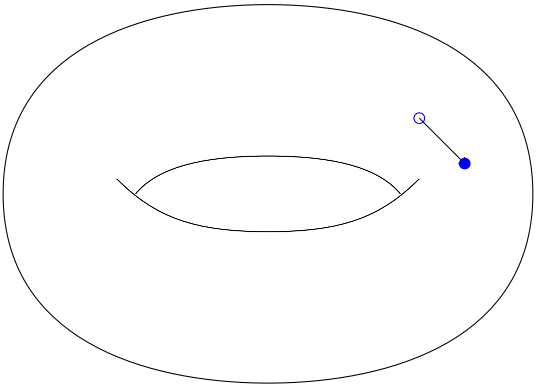

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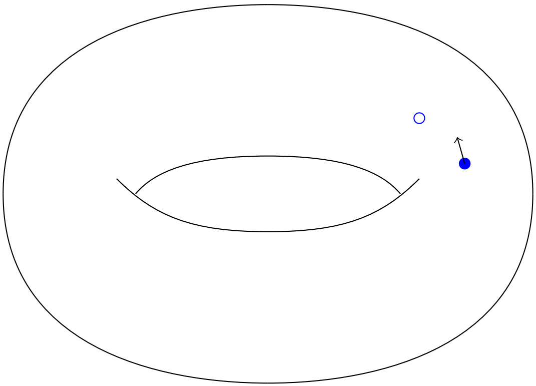

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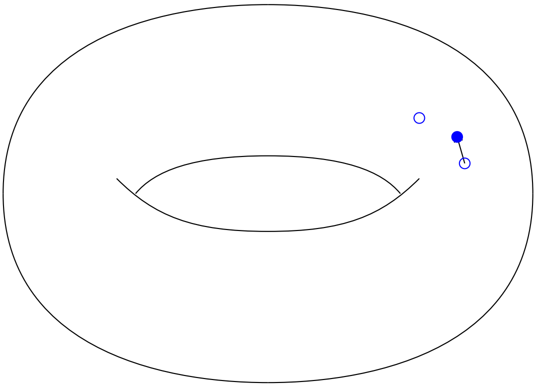

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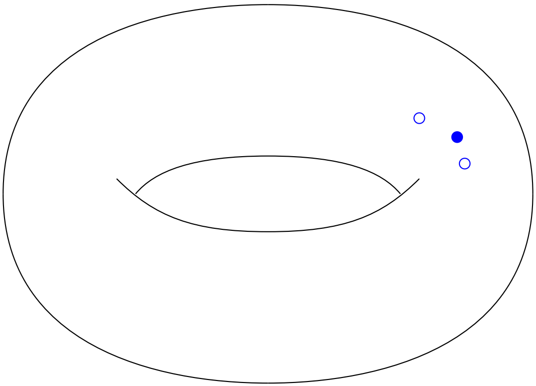

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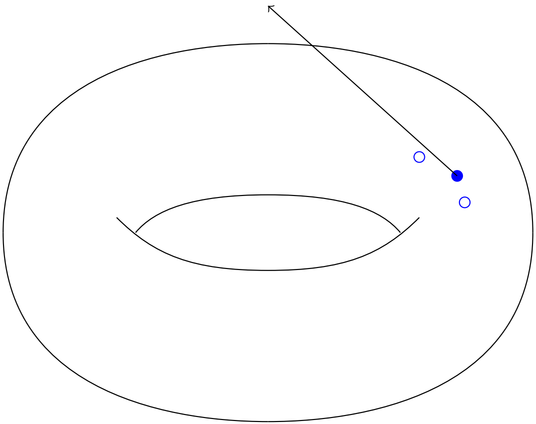
Take step size $h \to 0$, to get a continuous random walk $\{Y_t\}_{t\geq 0}$.

Generating random walks - Lévy flights



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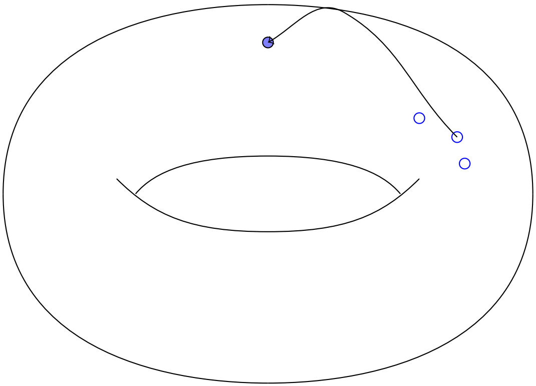
For any step: GLOBAL geometry

Generating random walks - Lévy flights



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Probability density functions of random walks - Infinitesimal generators

A different way to look at random walks: their probability density distribution

Lévy flight

Brownian motion

Probability density functions of random walks - Infinitesimal generators

A different way to look at random walks: their probability density distribution

Brownian motion

Let X_t be a Brownian motion starting at $x_0 \in M$, then the probability density distribution $v(t,x)$ of X_t is given by the fundamental solution of the heat equation

$$\frac{\partial v(t,x)}{\partial t} = \Delta_g v(t,x), \quad v(0,x) = \delta_{x_0}(x)$$

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$$\frac{\partial w(t,x)}{\partial t} = \mathcal{A} w(t,x), \quad w(0,x) = \delta_{x_0}(x)$$

Probability density functions of random walks - Infinitesimal generators

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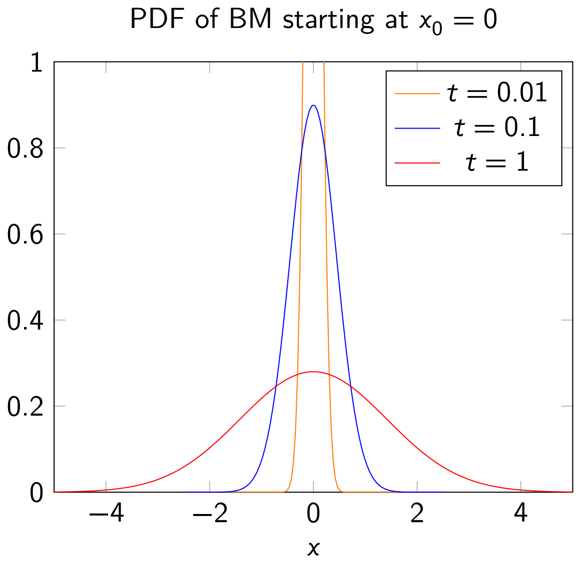
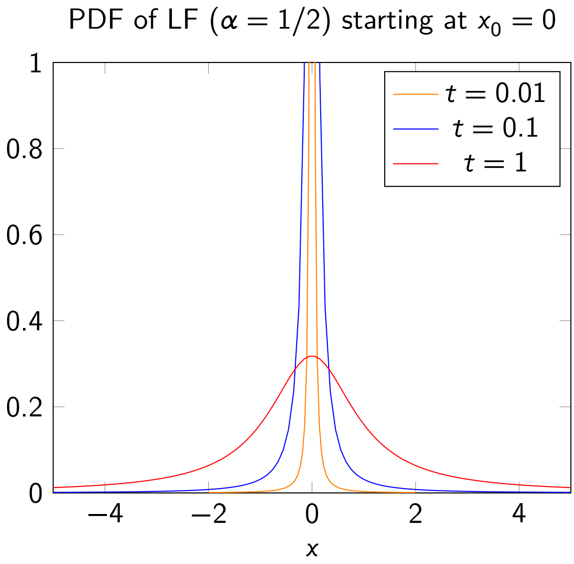
$\mathcal{A}$ is *not* always $-(-\Delta_g)^\alpha$!

Probability density functions of random walks in $\mathbb{R}$





Infinitesimal generators of Lévy flights on spheres

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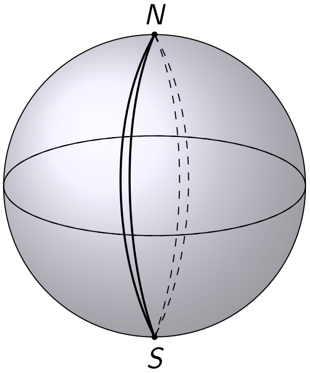

Figure 5: Two close-by "straight lines" passing through the North Pole focus back at the South Pole.

Infinitesimal generators of Lévy flights on spheres

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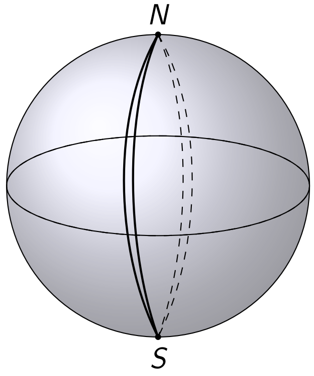
Theorem (Chaubet-Guedes Bonthonneau-Lefeuvre-Tzou). *On the sphere S^n, the infinitesimal generator $\mathcal{A}$ is the sum*

$$\mathcal{A} = -(-\Delta_g)^\alpha + \mathcal{A}_{-1}\mathcal{J}$$

with $\mathcal{J}u(x) = u(-x)$.



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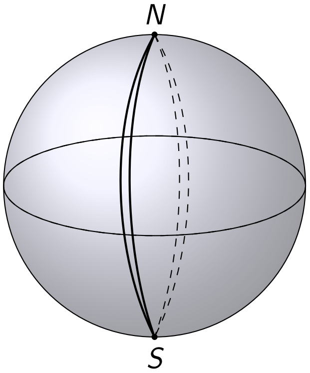
If $u(x)$ is not smooth at p, then $(\mathcal{A}u)(x)$ is not smooth at $\{p, -p\}$.



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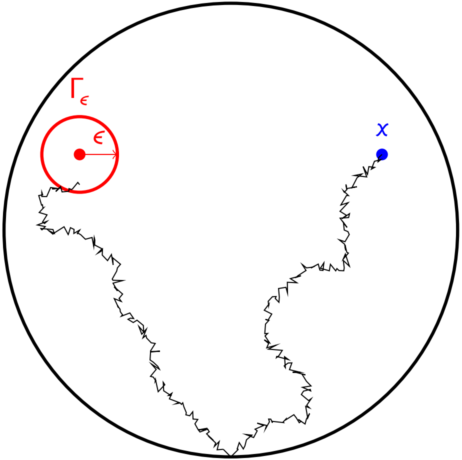
Narrow capture problems with infinitesimal generators

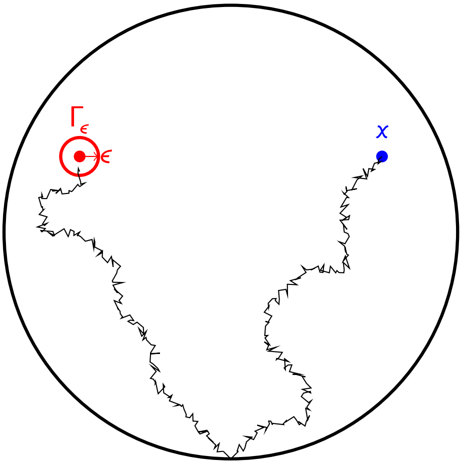


Let X_t a random walk starting at x with infinitesimal generator $\mathcal{A}$. Denote:

$$u_\epsilon(x) := \mathbb{E}[t : X_t \text{ in } \Gamma_\epsilon \text{ for the first time}]$$

Mean first capture times

Narrow capture problems with infinitesimal generators



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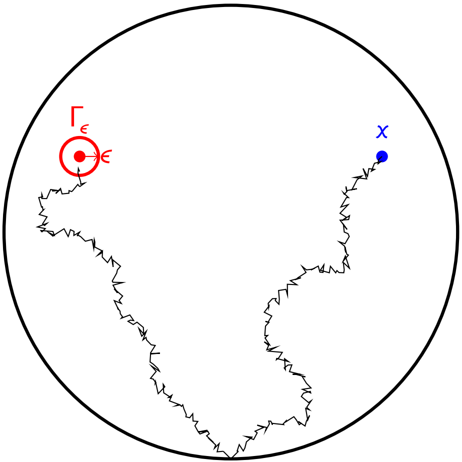

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The function $u_\epsilon(x)$ satisfies the boundary value problem

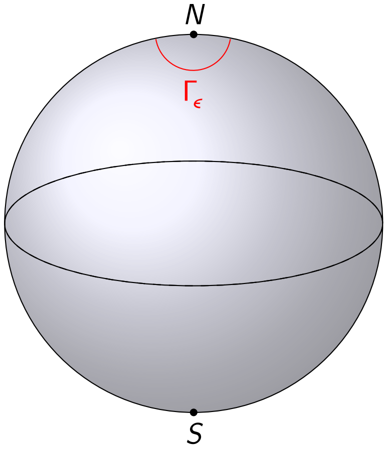
$$\mathcal{A}u_\epsilon(x) = -1 \quad u_\epsilon|_{\Gamma_\epsilon} = 0$$

with the possibility of additional conditions (e.g. reflecting outer boundary).

Deviation from the mean - round sphere

Theorem (Chaubet-Guedes Bonthonneau-Lefeuvre-Tzou). *Let $M = S^2$ the round sphere*

- *Lévy flight process of order $\alpha < 1/4$*



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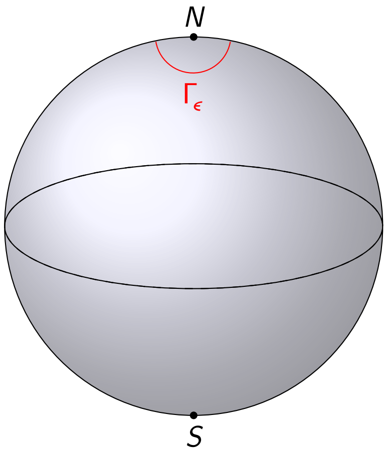

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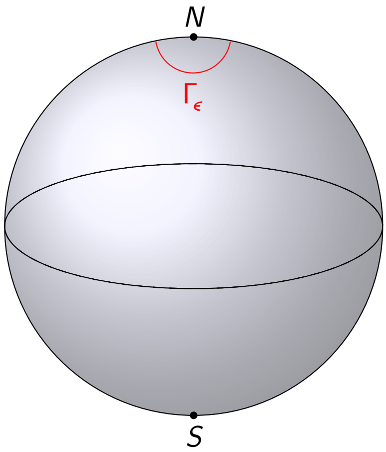
- *Spacial mean*

$$\bar{u}_\epsilon = \frac{1}{|M|}\int_M u_\epsilon(x)dx$$



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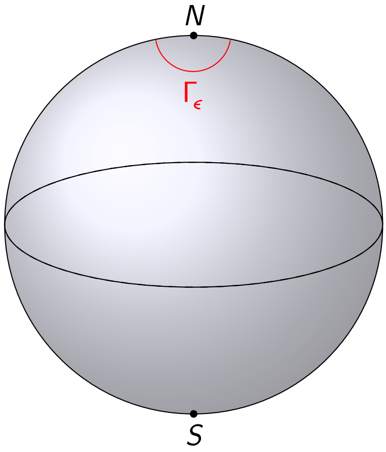
- *Target* $\Gamma_\epsilon = B_\epsilon(N)$
- *Spacial mean*
$$\overline{u}_\epsilon = \frac{1}{|M|}\int_M u_\epsilon(x)dx$$
- *As* $\epsilon \to 0$,
$$|u_\epsilon(x) - \overline{u}_\epsilon| \quad \textit{is bounded}$$

for $x \notin \{N, S\}$, *but*

$$|u_\epsilon(S) - \overline{u}_\epsilon| = C_{n,\alpha}\epsilon^{-1+4\alpha} + o(\epsilon^{-1+4\alpha})$$



Deviation from the mean - Zoll sphere

Theorem (Chaubet-G-Tzou, work in progress)**.** *Let M be a smooth Riemannian manifold homeomorphic to S^2 such that all geodesics are closed (and with the same period)*

- *Lévy flight process of order $\alpha < 1/4$*

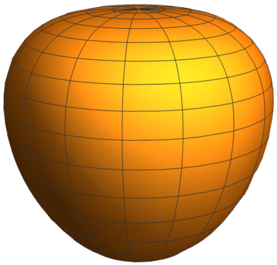


Figure 6: A Zoll manifold of revolution

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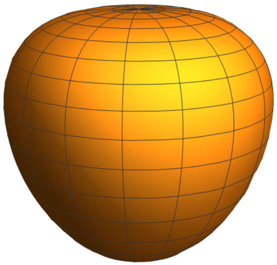

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$$|u_\epsilon(x) - \overline{u}_\epsilon| \quad \text{is bounded}$$

for $x \notin \{p_0, q_1, \ldots q_n\}$, but for $y \in \{q_1, \ldots, q_n\}$

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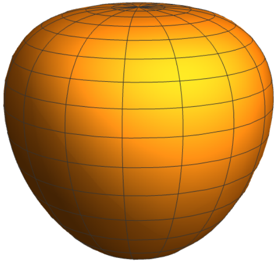

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Deviation from the mean - Zoll sphere

Theorem (Chaubet-G-Tzou, work in progress, part 2). *Let M be a smooth Riemannian manifold homeomorphic to S^2 such that all geodesics are closed (and with the same period)*

- *Lévy flight process of order $\alpha < 1/8$*

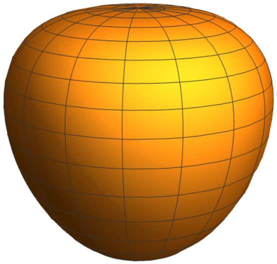
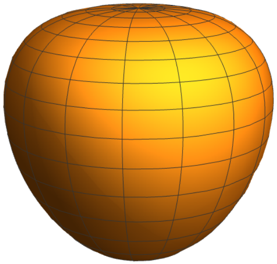


Figure 7: A Zoll manifold of revolution

Deviation from the mean - Zoll sphere

Theorem (Chaubet-G-Tzou, work in progress, part 2). *Let M be a smooth Riemannian manifold homeomorphic to S^2 such that all geodesics are closed (and with the same period)*

- *Lévy flight process of order* $\alpha < 1/8$
- *There is a curve* $\mathcal{C}$ *such that* $\{q_1, \ldots, q_n\}$ *are the singular points of the curve* $\mathcal{C}$



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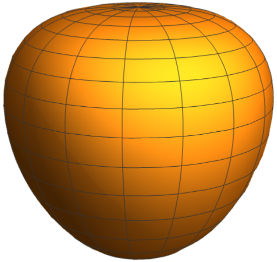
- *Lévy flight process of order $\alpha < 1/8$*
- *There is a curve $\mathcal{C}$ such that $\{q_1, \dots, q_n\}$ are the singular points of the curve $\mathcal{C}$*
- *If $z \in \mathcal{C} \setminus \{q_1, \dots, q_n\}$, then*

$$|u_\epsilon(z) - \overline{u}_\epsilon| = C'_{n,\alpha}\epsilon^{-1/2+4\alpha} + o(\epsilon^{-1/2+4\alpha})$$



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