STAMPS PRESENTATION

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Abstract. We discuss an easy proof to the Schwartz kernel theorem.

Today, we will be discussing one of analysis' most important theorems. For an embarrassingly long time, I was fully aware of the statement of this theorem, but could not really grasp the proof. However recently, I became aware of T. Lefeuvre's book [Lef25], in which the proof to this theorem is really well explained, and it is the main reference I will be using today.

Informally, the statement is the following:

Theorem. *Any continuous linear operator A between function spaces (such as $L^2(\mathbb{R}^n)$ or $\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n)$) can uniquely be expressed as an integral operator with an associated integral kernel K_A.*

Conversely, for every integral kernel K_A, there is a unique associated continuous linear operator A between function spaces.

Before I give a couple of definitions, the correct statement, and the proof, I will give a couple of examples.

Example 1. The identity operator

$$\mathbb{1} : \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n) \to \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n) : \varphi \mapsto \varphi$$

has the associated integral kernel $\delta(x-y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$. Indeed, for every $\varphi \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ we have

$$(\mathbb{1}\varphi)(x) = \int_{\mathbb{R}^n} \delta(x-y)\varphi(y)\ dy\ .$$

Example 2. The Laplacian $-\Delta := -\sum_{i=1}^n \partial_{x_i}$ has the associated integral kernel is the sum of the second partial derivatives of $\delta(x-y)$. In a different way,

$$\Delta\varphi(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}|\xi|^2\ d\xi\ \varphi(y)\ dy = \int_{\mathbb{R}^n} K_\Delta(x,y)\varphi(y)\ dy\ .$$

Exercise: show that these correspond.

To now formalise the statement, and to give a proof, I will need to define some function spaces.

Definition 1. Let

(1) $\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ to be the space of smooth functions with compact support (i.e. smooth functions which are zero outside of a compact set)

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(2) For all $s \in \mathbb{R}$, the space $H^s(\mathbb{R}^n)$ to be the completion of $\mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n)$ with respect to the norm
$$\|f\|^2_{H^s} = \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \, d\xi \quad \text{where } \langle \xi \rangle := (1 + |\xi|^2)^{1/2} \ .$$
These spaces are called Sobolev spaces.

(3) For all $l \in \mathbb{R}$, the space $H^{s,l}(\mathbb{R}^n) := \langle x \rangle^{-l} H^s(\mathbb{R}^n)$ to be the space of functions $f = \langle x \rangle^{-l} \tilde{f}$ for some $\tilde{f} \in H^s(\mathbb{R}^n)$. The natural norm is given by
$$\|f\|_{s,l} = \|f\|_{H^{s,l}(\mathbb{R}^n)} := \|\langle x \rangle^l f\|_{H^s(\mathbb{R}^n)} \ .$$
These spaces are sometimes called the scattering Sobolev spaces or the symmetrically global spaces.

(4) The Schwartz space $\mathcal{S}(\mathbb{R}^n) = \bigcap_{s,l} H^{s,l}(\mathbb{R}^n)$, and its dual, the tempered distributions $\mathcal{S}'(\mathbb{R}^n) = \bigcup_{s,l} H^{s,l}(\mathbb{R}^n)$

It turns out that for every $s, l \in \mathbb{R}$ the spaces $H^{s,l}(\mathbb{R}^n), H^{-s,-l}(\mathbb{R}^n)$ are dual to each other under the extension of the pairing
$$\mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n) \times \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n) \to \mathbb{C} : (\varphi, \psi) \mapsto \int_{\mathbb{R}^n} \varphi(x)\psi(x) \, dx$$
to $H^{s,l}(\mathbb{R}^n) \times H^{-s,-l}(\mathbb{R}^n) \to \mathbb{C}$.

Furthermore, there is a very important embedding:
$$H^m(\mathbb{R}^n) \hookrightarrow C^0_{\text{bounded}}(\mathbb{R}^n) \quad \text{if } m > n/2 \ .$$

With these function spaces in hand, we can state and prove the Schwartz kernel theorem.

Theorem 1 (Laurent Schwartz, 1952)**.** *Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$ be a continuous linear operator. Then, there exists a unique distribution $K_A \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n), \psi \in \mathcal{S}(\mathbb{R}^m)$ the relation*
$$(A\varphi, \psi) = (K_A, \psi \otimes \varphi)" = " \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} K(x,y)\varphi(y) \, dy \psi(x) \, dx \ .$$
Conversely, for any distribution $K_A \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$ defines a unique continuous linear operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$.

Proof. With the Stone-Weierstrass approximation Theorem (polynomials are dense inside of the smooth functions), it follows that $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{S}(\mathbb{R}^n)$ is dense inside $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$. Hence, if $(K_A, \psi \otimes \varphi) = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n), \psi \in \mathcal{S}(\mathbb{R}^m)$, then $K_A = 0$ from which it follows that $A = 0$, which shows the reverse implication.

To show the forward implication, if $A : L^2(\mathbb{R}^n) \to H^m(\mathbb{R}^m) \hookrightarrow C^0_{\text{bounded}}(\mathbb{R}^m)$, then for all $x \in \mathbb{R}^m$, the linear functional
$$T_x : L^2(\mathbb{R}^n) \to \mathbb{C} : \varphi \mapsto (A\varphi)(x)$$
is continuous. Hence, by the Riesz representation theorem, there exists a continuous map $t_x \in C^0_{\text{bounded}}(\mathbb{R}^m, L^2(\mathbb{R}^n))$ such that
$$(A\varphi)(x) = T_x(\varphi) = \langle t_x, \varphi \rangle = \int_{\mathbb{R}^n} t_x(y)\varphi(y) \, dy \ .$$
The function $(x, y) \mapsto K(x, y) := t_x(y)$ is the Schwartz kernel of the map A.

Now, we note that for any bounded (i.e. continuous) linear operator
$$A : H^{s,l}(\mathbb{R}^n) \to H^{-s,-l}(\mathbb{R}^m)$$
the operator
$$A' := \langle x\rangle^{-l}\langle\Delta\rangle^{-(s+m)/2}A\langle y\rangle^{-l}\langle\Delta\rangle^{-s/2} : L^2(\mathbb{R}^n) \to H^m(\mathbb{R}^m)\ ,$$
has some Schwartz kernel $K'(x,y)$, thus we find that
$$\begin{aligned}\langle A\varphi,\psi\rangle &= \langle A'\langle y\rangle^l\langle\Delta\rangle^{s/2}\varphi, \langle x\rangle^l\langle\Delta\rangle^{(s+m)/2}\psi\rangle\\ &= \langle K', \langle x\rangle^l\langle\Delta\rangle^{(s+m)/2}\psi \otimes \langle y\rangle^l\langle\Delta\rangle^{s/2}\varphi\rangle\\ &= \langle(\langle x\rangle^l\langle\Delta\rangle^{(s+m)/2} \otimes \langle y\rangle^l\langle\Delta\rangle^{s/2})K', \psi\otimes\varphi\rangle\\ &= \langle K, \psi\otimes\varphi\rangle\end{aligned}$$
with
$$K(x,y) := (\langle x\rangle^l\langle\Delta\rangle^{(s+m)/2} \otimes \langle y\rangle^l\langle\Delta\rangle^{s/2})K'(x,y)\ .$$
Note finally, that due to the definition of Schwartz spaces and tempered distributions, any continuous linear map $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m)$ is actually a bounded linear map between Sobolev spaces $A : H^{s,l}(\mathbb{R}^n) \to H^{-s,-l}(\mathbb{R}^m)$ for sufficiently large $s,l \in \mathbb{R}$. □

References

[Lef25] Thibault Lefeuvre, *Microlocal analysis in hyperbolic dynamics and geometry*, Cours Spécialisés, vol. 32, Société Mathématique de France, Paris, 2025. With a contributed chapter by Yann Chaubet. MR5022124